Distributed Spanner with Bounded Degree for Wireless Ad Hoc Networks

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Abstract

In this paper, we propose a new distributed algorithm that constructs a sparse spanner subgraph of the unit disk graph efficiently for wireless ad hoc networks. It maintains a linear number of links while still preserving some power-efficient routes for any pair of nodes. We prove that this new topology not only has a bounded node degree k but also has a bounded power stretch factor 2 in civilized graph. Finally, we give some simulation results to verify its performance in practice and conclude the paper with the discussion of possible future works.

1 Introduction

Wireless ad hoc networks has various applications in many area and has drawn considerable attentions both from network engineers and theoretical researchers. One of the major concern in designing wireless ad hoc networks is to save the energy consumed as the wireless nodes are often powered by batteries only. In addition, the scalability is crucial for network operations as every node has limited resource such as memory. One effective approach to cope these constraints is to maintain only a linear number of links using a *localized* construction method. Here a distributed algorithm constructing a graph G is a *localized algorithm* if every node u can exactly decide all edges incident on u based only on the information of all nodes within a constant hops of u. At the same time, the sparseness property should not compromise too much on the power consumptions on communications. In this paper, we study how to construct a sparse spanner efficiently for a set of static wireless nodes such that there is a power-efficient unicast route in the constructed network topology for any given pair of nodes.

We consider a wireless ad hoc network consisting of a set V of wireless nodes distributed in a two-dimensional plane. For the sake of simplicity, we assume that the nodes are static or static in a reasonable time period. In the most common power-attenuation model, the power needed to support a link uv is $\|uv\|^{\beta}$, where $\|uv\|$ is the Euclidean distance between u and v, and β is a real constant between 2 and 4 dependent on the wireless transmission environment. By a proper scaling, we assume that all nodes have the maximum transmission range equal to one unit. These wireless nodes define a *unit disk graph* $UDG(V)$ in which there is an edge between two nodes if and only if their Euclidean distance is at most one. The size of the unit disk graph could be as large as the square order of the number of network nodes.

A trade-off can be made between the sparseness of the topology and the power efficiency. The power efficiency of any spanner is measured by its power stretch factor, which is defined as the maximum ratio of the minimum power needed to support the connection of two nodes in this spanner to the least necessary in the unit disk graph. Recently, Wattenhofer *et al.* [14] proposed a two-phased power efficient network construction consisting of a variation of the Yao structure followed by a variation of the Gabriel graph. In [10], Li *et al.* studied the power efficiency property of several well-known proximity graphs including the relative neighborhood graph,the Gabriel graph and the Yao graph. These graphs are sparse and can be constructed locally in an efficient way. They showed that the power stretch factor of the Gabriel graph is always one, and the power stretch factor of the Yao graph is bounded from above by a real constant while the power stretch factor of the relative neighborhood graph could be as large as the network size minus one. Notice that all of these graphs do *not* have constant bounded node degrees, which can cause *large* overhead at some nodes in wireless network. They further suggested to use another sparse topology, called *Yao and Sink*, that has both a constant bounded node degree and a constant bounded power stretch factor. An efficient but complicated localized algorithm was presented for constructing this topology.

In [11], Li *et al.* also defined another structure named *Yao plus reverse Yao*, denoted by $\overrightarrow{YY}_k(V)$ hereafter, which has a bounded node degree too. Their experiments showed that it has a small stretch factor in practice. However they did not give a proof of its spanner property. In this paper, we review the algorithm to construct $\overrightarrow{YY}_k(V)$ and show that it is power-efficient in civilized graph and each node has a bounded node degree.

The rest of the paper is organized as follows. In Section 2, we first give some definitions and review some results related to the distributed spanner for wireless networks. In Section 3, we review the method to construct $\overrightarrow{YY}_k(V)$, and prove the correctness of the method, *i.e.*, the resulting topology has a bounded power stretch factor and a bounded node degree. Its practical performance is studied in Section 4. We conclude our paper in Section 5 by discussing some possible future works.

2 Preliminaries

In this section, we will give some geometry definitions and notations that will be used in our presentation later.

2.1 Spanner and Power Stretch Factor

Constructing a spanner of a graph has been well studied by computational geometry community [1, 2, 4, 6, 12, 15]. Let $\pi_G(u, v)$ be the shortest path connecting u and v in a graph G. Then a graph H is a spanner of G if there exists a constant t such that the length of $\pi_H(u, v)$ is no more than t times the length of $\pi_G(u, v)$ for any two nodes u and v. The constant t is called the length stretch factor[1]. However for wireless networks, we pay more attention on the power consumptions, the following definition of power stretch factor was introduced in [11].

Consider any unicast path $\pi(u, v)$ in G (could be directed) from a node $u \in V$ to another node $v \in V$:

$$\pi(u, v) = v_0 v_1 \cdots v_{h-1} v_h,$$

where $u = v_0$, $v = v_h$ and h is the number of hops of the path π. The total *transmission power* $p(\pi)$ consumed by this path π is defined as

$$p(\pi) = \sum_{i=1}^{h} \|v_{i-1} v_i\|^{\beta}$$

Let $p_G(u, v)$ be the least energy consumed by all paths connecting nodes u and v in G. The path in G connecting u, v and consuming the energy $p_G(u, v)$ is called the *least-energy path* in G for u and v.

Let H be a subgraph of G. Its *power stretch factor* with respect to G is then defined as

$$\rho_H(G) = \max_{u,v \in V} \frac{p_H(u, v)}{p_G(u, v)}$$

[1]Some researchers call it *dilation ratio*, *spanning ratio*.

If G is a unit disk graph, we use $\rho_H(V)$ instead of $\rho_H(G)$. For any n, let

$$\rho_H(n) = \sup_{|V|=n} \rho_H(V).$$

When the graph H is clear from the context, it is dropped from notation.

Notice that, the power stretch factor has the following monotonic property [10, 11].

Lemma 1 *If $H_1 \subset H_2 \subset G$ then the power stretch factors of H_1 and H_2 satisfy $\rho_{H_1}(G) \geq \rho_{H_2}(G)$.*

2.2 Well-known Structures

Various proximity subgraphs of the unit disk graph can be defined and be used in the topology control or other applications for wireless networks [3, 7, 8, 10, 13, 14].

The *relative neighborhood graph*, denoted by $RNG(V)$, consists of all edges uv such that $\|uv\| \leq 1$ and there is no point $w \in V$ such that $\|uw\| < \|uv\|$, and $\|wv\| < \|uv\|$. The *Gabriel graph*, denoted by $GG(V)$, consists of all edges uv such that $\|uv\| \leq 1$ and the open disk using uv as diameter does not contain any node from V. The *Yao graph* with an integer parameter $k \geq 6$, denoted by $\overrightarrow{YG}_k(V)$, is defined as follows. At each node u, any k equal-separated rays originated at u defined k cones. In each cone, choose the closest node v to u with distance at most one, if there is any, and add a directed link $\overrightarrow{uv}$. Ties are broken arbitrarily. Let $YG_k(V)$ be the undirected graph by ignoring the direction of each link in $\overrightarrow{YG}_k(V)$. See Figure 1 for an illustration.

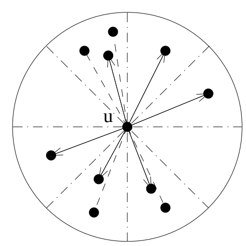

Figure 1. Neighbors of u in Yao graph.

These graphs extend the conventional definitions of corresponding ones for the completed Euclidean graph. It is well-known that $RNG(V)$ is a subgraph of $GG(V)$ and $YG_k(V)$. In addition, all these graphs contain the Euclidean minimum spanning tree as a subgraph. These graphs are sparse: $|RNG(V)| \leq 3n - 10$, $|GG(V)| \leq 3n - 8$, and $|\overrightarrow{YG}_k(V)| \leq kn$.

Bose *et al.* [2] showed that the length stretch factor of $RNG(V)$ is $\Theta(n-1)$ and the length stretch factor of $GG(V)$ is $\Theta(\frac{4\pi\sqrt{2n-4}}{3})$. Several papers showed that the Yao graph $\overrightarrow{YG}_k(V)$ has length stretch factor at most $\frac{1}{1-2\sin\frac{\pi}{k}}$. Recently, Li *et al.* [10] studied the power efficiency property of these well-known proximity graphs. They showed that the power stretch factor of the Gabriel graph is always 1, and the power stretch factor of the Yao graph is bounded from above by a real constant $\frac{1}{1-(2\sin\frac{\pi}{k})^\beta}$ while the power stretch factor of the relative neighborhood graph could be as large as $n-1$. All these structures can be constructed locally.

2.3 Bounded Degree?

The sparseness of these well-known proximity graphs implies that the average node degree is bounded by a constant. However the maximum degree could be as large as $n-1$ as shown in Figure 2. The instance consists of $n-1$ points lying on the unit circle centered at a node $u \in V$. Then each edge uv_i belongs to the $RNG(V)$, $GG(V)$ and $\overrightarrow{YG}_k(V)$. Thus, some node may have a very large degree (in-degree for $\overrightarrow{YG}_k(V)$) in $RNG(V)$, $GG(V)$ and $\overrightarrow{YG}_k(V)$, although $\overrightarrow{YG}_k(V)$ has a bounded out-degree k for each node.

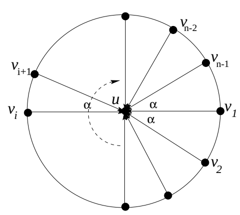

Figure 2. u **has degree (or in-degree)** $n-1$**.**

Because wireless nodes have limited resources, we prefer the node degree is bounded by a constant. Unbounded degree (or in-degree) at node u will often cause large overhead at u. On the other hand, bounded degree will also give us advantages when apply several routing algorithms. Therefore it is often imperative to construct a sparse network topology such that both the in-degree and the out-degree are bounded by a constant while it is still power-efficient.

Arya *et al.* [1] had given an ingenious technique to generate a bounded degree graph with a constant length stretch factor. In [10], the authors apply the same technique to construct a sparse network topology with a bounded degree and a bounded power stretch factor. The technique is to replace the directed star consisting of all links towards a node u by a directed tree $T(u)$ with u as the sink. Tree $T(u)$ is constructed recursively. See [10] for more detail. Figure 3 (a) illustrates a directed star centered at u and Figure 3 (b) shows the directed tree $T(u)$ constructed to replace the star.

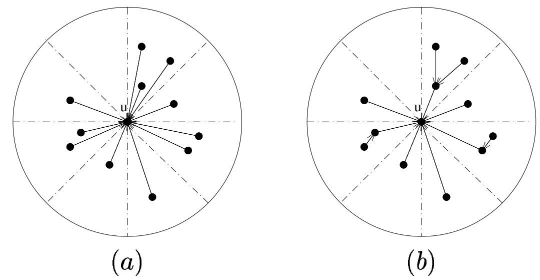

Figure 3. (a) Star formed by links towards to u**. (b) Directed tree** $T(u)$ **sinked at** u**.**

The union of all trees $T(u)$ is called the *sink structure* $\overrightarrow{YG}_k^*(V)$. They [10] proved that its power stretch factor is at most $(\frac{1}{1-(2\sin\frac{\pi}{k})^\beta})^2$, and its in-degree is bounded by $k(k+1)$ while the maximum out-degree is k. However, the construction of $\overrightarrow{YG}_k^*(V)$ is actually more complicated and the performance gain compared with $\overrightarrow{YY}_k(V)$ is not so obvious in practice as shown by our experimental results.

2.4 Civilized Graph

When we prove the spanner property of the new structure, we consider it in the *civilized unit disk graph* instead of general unit disk graph. Here *civilized graph* means that the distance between any two nodes in this graph is larger than a constant λ. In [5], they call the civilized unit disk graph as the λ*-precision unit disk graph*. Notice the wireless devices in wireless network can not be too close, so it is reasonable to model the wireless ad hoc networks as a *civilized unit disk graph*.

3 New Sparse Spanner

We begin this section by reviewing the algorithm constructing $\overrightarrow{YY}_k(V)$. Then we prove the correctness of this algorithm by showing the connectivity and spanner property of the new topology.

3.1 Constructing $YY_k(V)$

Assume that each node v_i of V has a unique identification number $ID(v_i) = i$. The identity of a directed link $\overrightarrow{uv}$ is defined as

$$ID(\overrightarrow{uv}) = (||uv||, ID(u), ID(v)).$$

Then we can order all directed links (at most $n(n-1)$ such links) in an increasing order of their identities. Here the identities of two links are ordered based on the following rule: $ID(\overrightarrow{uv}) > ID(\overrightarrow{pq})$ if

1. $||uv|| > ||pq||$ or

2. $||uv|| = ||pq||$ and $ID(u) > ID(p)$ or

3. $||uv|| = ||pq||$, $u = p$ and $ID(v) > ID(q)$.

Correspondingly, the rank, denoted by $rank(\overrightarrow{uv})$, of each directed link $\overrightarrow{uv}$ is its order in the sorted directed links. Notice that, we actually only have to consider the links with length no more than one.

Algorithm: Constructing-YY

1. Each node u divides the space by k equal-sized cones centered at u. We generally assume that the cone is half open and half-close. Node u chooses a node v from each cone so the directed link $\overrightarrow{uv}$ has the smallest $ID(\overrightarrow{uv})$ among all directed links $\overrightarrow{uv}_i$ with v_i in that cone, if there is any. Let $\overrightarrow{YG}_k(V)$ be the union of all chosen directed links.

2. Node u chooses a node v from each cone, if there is any, so the directed link $\overrightarrow{vu}$ has the smallest $ID(\overrightarrow{vu})$ among all directed links computed in the first step in that cone. See Figure 4.

3. The union of all chosen directed links in the second step is the final network topology, denoted by $\overrightarrow{YY}_k(V)$. If the link directions are ignored, the graph is denoted as $YY_k(V)$.

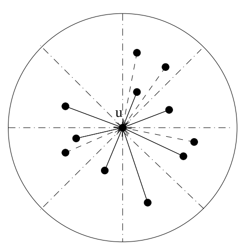

Figure 4. Node u chooses the shortest links.

It is obvious that both the out-degree and in-degree of a node in $\overrightarrow{YY}_k(V)$ are bounded by k. This implies that $\overrightarrow{YY}_k(V)$ is a sparse graph. From the construction algorithm, we also know $\overrightarrow{YY}_k(V)$ is a subgraph of $\overrightarrow{YG}^*_k(V)$, because all the links which are selected by node u in the second step are included by the directed tree $T(u)$ built at node u in the *sink structure* $\overrightarrow{YG}^*_k(V)$.

3.2 Correctness

3.2.1 Connectivity

Theorem 2 *The directed graph $\overrightarrow{YY}_k(V)$ is strongly connected if $UDG(V)$ is connected and $k > 6$.*

PROOF. Notice that it is sufficient to show that there is a directed path from u to v for any two nodes u and v with $||uv|| \leq 1$. Notice that the Yao graph $\overrightarrow{YG}_k(V)$ is strongly connected. Therefore, we only have to show that for any directed link $\overrightarrow{uv}$ in $\overrightarrow{YG}_k(V)$, there is a directed path from u to v in $\overrightarrow{YY}_k(V)$. We prove the claim by induction on the ranks of all directed links in $\overrightarrow{YG}_k(V)$.

If the directed link $\overrightarrow{uv}$ has the smallest rank among all links of $\overrightarrow{YG}_k(V)$, then $\overrightarrow{uv}$ will obviously survive after the second step. So the claim is true for the smallest rank.

Assume that the claim is true for all links in $\overrightarrow{YG}_k(V)$ with rank at most r. Then consider a directed link $\overrightarrow{vu}$ in $\overrightarrow{YG}_k(V)$ with $rank(\overrightarrow{vu}) = r + 1$ in $\overrightarrow{YG}_k(V)$. If $\overrightarrow{vu}$ survives in the second phase, then the claim holds. Otherwise, $\overrightarrow{vu}$ can only be removed by the node u in the second phase. Then there must exist a directed link $\overrightarrow{wu}$ survived with a smaller identity in the same cone as $\overrightarrow{vu}$. In addition, the angle $\angle wuv$ is less than $\theta = \frac{2\pi}{k}$. Here $(||wu||, ID(w), ID(u)) < (||vu||, ID(v), ID(u))$. Therefore $||wu|| \leq ||vu||$. Because $\angle wuv < \frac{2\pi}{k}$, we have $||wv|| < ||uv||$. Consequently, the identity $(||vw||, ID(v), ID(w))$ of the directed link $\overrightarrow{vw}$ is less than that of the directed link $\overrightarrow{vu}$, which is $(||vu||, ID(v), ID(u))$. Notice that here the directed link $\overrightarrow{vw}$ is not guaranteed to be in $\overrightarrow{YG}_k(V)$ and our induction is for all directed links in $\overrightarrow{YG}_k(V)$. So we can not directly use the induction. There are two cases here.

Case 1: the link $\overrightarrow{vw}$ is in $\overrightarrow{YG}_k(V)$. Then by induction, there is a directed path $v \rightsquigarrow w$ from v to w after the second phase. Consequently, there is a directed path (concatenation of the path $v \rightsquigarrow w$ and the link $\overrightarrow{wu}$) from v to u after the second phase.

Case 2: the link $\overrightarrow{vw}$ is not in $\overrightarrow{YG}_k(V)$. Then we know that there is a directed path $\pi_{\overrightarrow{YG}_k}(v, w) = q_1 q_2 \cdots q_h$ from v to w in $\overrightarrow{YG}_k(V)$, where $q_1 = v$ and $q_h = w$. Using the same proof technique, we can prove that each directed link $q_i q_{i+1}$, $1 \leq i < h$, in $\pi_{\overrightarrow{YG}_k}(v, w)$ has a smaller rank than $\overrightarrow{vw}$, which is r. Here $rank(q_1 q_2 = vq_2) < rank(v, w)$ because the selection method in the first step. And $rank(q_i q_{i+1}) < rank(v, w)$, $1 < i < h$, because $||q_i q_{i+1}|| \leq ||q_i w|| < ||q_{i-1} w|| < \cdots < ||q_1 w|| = ||vw||$. Then for each link in $q_i q_{i+1}$ in $\pi_{\overrightarrow{YG}_k}(v, w)$, there is a directed path $q_i \rightsquigarrow q_{i+1}$ survived in $\overrightarrow{YY}_k(V)$ after the second phase (this is proved by induction on the rank $rank(q_i q_{i+1})$). The the concatenation of all such paths $q_i \rightsquigarrow q_{i+1}$, $1 \leq i < h$, and the directed link $\overrightarrow{wu}$ forms a directed path from v to u in $\overrightarrow{YY}_k(V)$.

This finishes the proof of the strong connectivity theorem. ◻

3.2.2 Spanner Property

After the proof of its connectivity, we now prove that $\overrightarrow{YY}_k(V)$ is a spanner in civilized graph. Remember that in a civilized graph the distance between any two nodes is at least λ.

Theorem 3 *The power stretch factor of the directed topology $\overrightarrow{YY}_k(V)$ is bounded by a constant ρ in civilized graph.*

PROOF. We actually prove the following claims by induction on the rank of the directed links:

1. There is a constant $\delta > 1$, such that for any directed link $\overrightarrow{v_iv_j}$ in the graph $\overrightarrow{YG}_k(V)$, the least energy consumption path in $\overrightarrow{YY}_k(V)$ from v_i to v_j is no more than $\delta||v_iv_j||^\beta$. In other words, we show that
$$p_{\overrightarrow{YY}_k(V)}(v_i, v_j) \leq \delta||v_iv_j||^\beta.$$
2. There is a constant $\rho > \delta$, such that for any directed link $\overrightarrow{v_iv_j}$ not in the graph $\overrightarrow{YG}_k(V)$, the least energy consumption path in $\overrightarrow{YY}_k(V)$ from v_i to v_j is no more than $\rho||v_iv_j||^\beta$. In other words, we show that
$$p_{\overrightarrow{YY}_k(V)}(v_i, v_j) \leq \rho||v_iv_j||^\beta.$$

For the directed link with the rank one, it is in $\overrightarrow{YY}_k(V)$, therefore the first claim holds. Assume that the claims are true for all links with rank at most r. Then consider the directed link $\overrightarrow{ut}$ with rank $r+1$.

Case 1: link $\overrightarrow{ut}$ does not belong to $\overrightarrow{YG}_k(V)$. Then there is a directed path $\pi_{\overrightarrow{YG}_k}(u,t) = q_1q_2\cdots q_h$ from u to t in graph $\overrightarrow{YG}_k(V)$, where $q_1 = u$ and $q_h = t$. Let v be node q_2. Then we have

$$\begin{aligned} &rank(\overrightarrow{uv}) = (||\overrightarrow{uv}||, ID(u), ID(v)) \\ &< rank(\overrightarrow{ut}) = (||\overrightarrow{ut}||, ID(u), ID(t)) \end{aligned}$$

because of the selection method of the first step. Similarly,

$$\begin{aligned} &rank(\overrightarrow{vt}) = (||\overrightarrow{vt}||, ID(v), ID(t)) \\ &< rank(\overrightarrow{ut}) = (||\overrightarrow{ut}||, ID(u), ID(t)) \end{aligned}$$

because $||\overrightarrow{vt}|| < ||\overrightarrow{ut}||$. Then we can apply the induction on $\overrightarrow{uv}$ and $\overrightarrow{vt}$. Notice that here $\overrightarrow{vt}$ may not belong to $\overrightarrow{YG}_k(V)$. Consequently, we have

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq \delta||uv||^\beta, \quad p_{\overrightarrow{YY}_k(V)}(v,t) \leq \rho||vt||^\beta.$$

Therefore, we have

$$\begin{aligned} p_{\overrightarrow{YY}_k(V)}(u,t) &\leq p_{\overrightarrow{YY}_k(V)}(u,v) + p_{\overrightarrow{YY}_k(V)}(v,t) \\ &\leq \delta||uv||^\beta + \rho||vt||^\beta \end{aligned}$$

There are two subcases here. Either $\angle uvt$ is acute or not.

Subcase 1.1: the angle $\angle uvt$ is not acute. Then we have

$$||uv||^\beta + ||vt||^\beta \leq ||ut||^\beta.$$

It implies that

$$\delta||uv||^\beta + \rho||vt||^\beta < \rho(||uv||^\beta + ||vt||^\beta) \leq \rho||ut||^\beta$$

Consequently, we have

$$p_{\overrightarrow{YY}_k(V)}(u,t) \leq \delta||uv||^\beta + \rho||vt||^\beta < \rho||ut||^\beta$$

In other words, the claims hold for this subcase as long as $\delta < \rho$.

Subcase 1.2: the angle $\angle uvt$ is acute. Then we have

$$||vt|| \leq 2\sin\frac{\theta}{2}||ut|| = 2\sin\frac{\pi}{k}||ut||.$$

Consequently, we have

$$\begin{aligned} \delta||uv||^\beta + \rho||vt||^\beta &\leq \delta||ut||^\beta + \rho||vt||^\beta \\ &\leq \delta||ut||^\beta + \rho\left(2\sin\frac{\pi}{k}||ut||\right)^\beta \\ &= \left(\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta\right)||ut||^\beta \end{aligned}$$

Therefore, if

$$\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta \leq \rho$$

then we have

$$\begin{aligned} p_{\overrightarrow{YY}_k(V)}(u,t) &\leq \delta||uv||^\beta + \rho||vt||^\beta \\ &\leq \left(\delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta\right)||ut||^\beta \\ &\leq \rho||ut||^\beta \end{aligned}$$

In other words, the claims hold for this subcase.

Case 2: a link $\overrightarrow{uv}$ with rank $r+1$ does belong to $\overrightarrow{YG}_k(V)$. Then we know that there is a directed path $\pi_{\overrightarrow{YY}_k}(u,v) = v_1v_2\cdots v_h$ from u to v in $\overrightarrow{YY}_k(V)$, where $v_1 = u$ and $v_h = v$. Let $w = v_{h-1}$. If w is u then we have

$$p_{\overrightarrow{YY}_k(V)}(u,v) \leq ||uv||^\beta < \delta||uv||^\beta.$$

So the claims hold. Otherwise, because the directed links $\overrightarrow{uv}$ and $\overrightarrow{wv}$ are at a same cone centered at v, we have

$$rank(\overrightarrow{wv}) < rank(\overrightarrow{uv})$$

due to the selection method in the second phase. Notice that $\angle uvw < \frac{2\pi}{k}$, we have $||uw|| < ||uv||$. So we also have

$$rank(uw) < rank(\overrightarrow{uv}) = r + 1$$

Then by induction, we know that the least energy consumption path in $\overrightarrow{YY}_k(V)$ connecting u and w is bounded by $\rho||uw||^\beta$. Therefore there is a path in $\overrightarrow{YY}_k(V)$ from u to v with the least energy consumption at most $\rho||uw||^\beta + ||wv||^\beta$. Notice we want to show

$$p_{\overrightarrow{YY}_k(V)}(u,v) \le \delta||uv||^\beta.$$

Subcase 2.1: the angle $\angle uwv$ is not acute. Then we have

$$||uw||^\beta + ||wv||^\beta \le ||uv||^\beta.$$

It implies that

$$\begin{aligned} \rho||uw||^\beta + ||wv||^\beta &\le \rho(||uv||^\beta - ||wv||^\beta) + ||wv||^\beta \\ &= \rho||uv||^\beta - (\rho-1)||wv||^\beta \end{aligned}$$

For the property of civilized graph, we know $||wv||^\beta \ge \lambda^\beta$. Then we have

$$p_{\overrightarrow{YY}_k(V)}(u,v) \le \rho||uv||^\beta - (\rho-1)\lambda^\beta.$$

Therefore, if

$$\rho - \delta \le (\rho-1)\lambda^\beta$$

then we have

$$\begin{aligned} p_{\overrightarrow{YY}_k(V)}(u,v) &\le \rho||uv||^\beta - (\rho-1)\lambda^\beta \\ &\le \rho||uv||^\beta - (\rho-\delta) \\ &< \rho||uv||^\beta - (\rho-\delta)||uv||^\beta \\ &\le \delta||uv||^\beta \end{aligned}$$

In other words, the claims hold for this subcase.

Subcases 2.2: the angle $\angle uwv$ is acute. Then we have

$$||uw|| \le 2\sin\frac{\theta}{2}||uv|| = 2\sin\frac{\pi}{k}||uv||.$$

Consequently, we have

$$\begin{aligned} \rho||uw||^\beta + ||wv||^\beta &\le \rho||uw||^\beta + ||uv||^\beta \\ &\le \rho\left(2\sin\frac{\pi}{k}||uv||\right)^\beta + ||uv||^\beta \\ &= \left(\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1\right)||uv||^\beta \end{aligned}$$

Therefore, if

$$\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1 \le \delta$$

then we have

$$\begin{aligned} p_{\overrightarrow{YY}_k(V)}(u,t) &\le \rho||uw||^\beta + ||wv||^\beta \\ &\le \left(\rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1\right)||uv||^\beta \\ &\le \delta||uv||^\beta \end{aligned}$$

In other words, the claims hold for this subcase.

Consequently, if

$$\begin{cases} \delta + \rho\left(2\sin\frac{\pi}{k}\right)^\beta \le \rho, \\ \rho\left(2\sin\frac{\pi}{k}\right)^\beta + 1 \le \delta, \\ \rho - (\rho-1)\lambda^\beta \le \delta, \end{cases}$$

then the claims hold for all the cases. This will finish the proof of the theorem.

So now we consider whether there exist the constants ρ, δ, k and λ which make these conditions hold. First assume $\alpha = \left(2\sin\frac{\pi}{k}\right)^\beta$, we need

$$\begin{cases} \delta + \rho\alpha \le \rho, \\ \rho\alpha + 1 \le \delta, \\ \rho - (\rho-1)\lambda^\beta \le \delta, \end{cases}$$

to hold. If $1-\alpha > 0$ and $1-\lambda^\beta > 0$, we can transfer these conditions to

$$\frac{\delta}{1-\alpha} \le \rho \le \min\left(\frac{\delta-1}{\alpha}, \frac{\delta-\lambda^\beta}{1-\lambda^\beta}\right).$$

So for a given small λ, if we select k such that $\alpha = \left(2\sin\frac{\pi}{k}\right)^\beta < \lambda^\beta$ then the existence of δ and ρ is guaranteed. For example, we can choose $\alpha = \lambda^\beta/2$, then $\delta = \frac{2-\lambda^\beta}{2-2\lambda^\beta}$ and $\rho = \frac{\delta}{1-\alpha} = 2$. Then we can get the bounded stretch factor. ◻

Here we only prove the spanner property of $\overrightarrow{YY}_k(V)$ in civilized graph. Our experimental results show that this sparse topology has a small power stretch factor in practice (see Section 4). We conjecture that $\overrightarrow{YY}_k(V)$ also has a constant bounded power stretch factor theoretically in any general graph. The proof of this conjecture or the construction of a counter-example remain a future work.

4 Experiments

In this section we measure the performance of the new sparse and power efficient topology by conducting some experiments. In a wireless network, each node is expected to potentially send and receive messages from many nodes. Therefore an important requirement of such network is a strong connectivity. In Section 3, we show that $YY_k(V)$ is strongly connected if $UDG(V)$ is connected. So in our experiments, we randomly generate a set V of n wireless nodes and its $UDG(V)$, and test the connectivity of $UDG(V)$. If it is strongly connected, we construct different topologies from V. Then we measure the sparseness and the power efficiency of these topologies by the following criteria: the average and the maximum node degree, and the average and the maximum power stretch factor. In the

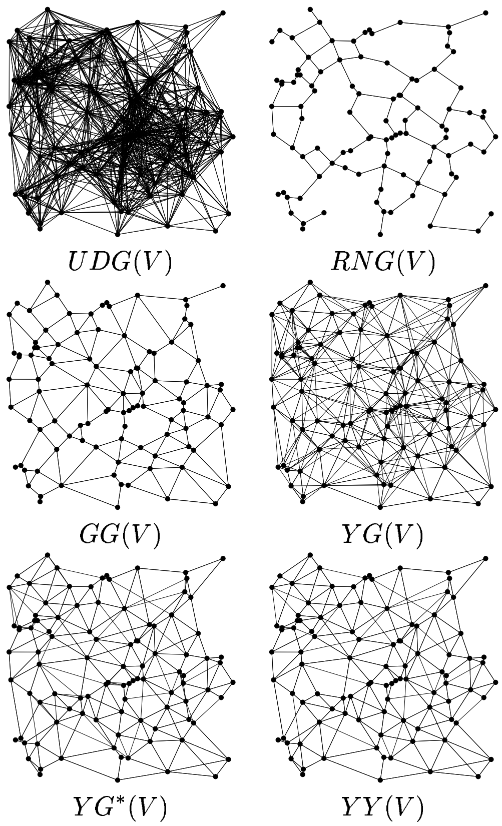

Figure 5. Different topologies from $UDG(V)$.

experimental results presented here, we generate 100 random wireless nodes in a 10×10 square; the number of cones is set to 8 when we construct $YG_k(V)$, ${YG_k}^*(V)$ and $YY_k(V)$; the power attenuation constant $\beta = 2$; the transmission range is set as $\sqrt{10}$. We generate 500 vertex sets V (each with 100 vertices) and then generate the graphs for each of these 500 vertex sets. The average and the maximum are computed over all these 500 vertex sets. Figure 5 gives all five different topologies for the unit disk graph illustrated by the first figure of Figure 5.

4.1 Node Degree

The node degree of the wireless networks should not be too large. Otherwise a node with a large degree has to communicate with many nodes directly. This increases the interference and collision, and increases the overhead at this node. The node degree should also not be too small either: a low node degree usually implies that the network has a lower fault tolerance and it also tends to increase the overall network energy consumption as longer paths may have to be taken. Thus the node degree is an important performance metric for the wireless network topology.

The node degrees of each topology are shown in Table 1. Here d_{avg}/d_{max} is the average/maximum node degree; I_{avg}/I_{max} is the average/maximum node in-degree; O_{avg}/O_{max} is the average/maximum node out-degree. Notice that for a directed graph, its I_{avg} equals to its O_{avg}. It shows that $YY_k(V)$ has a much less number of edges than $YG_k(V)$ and ${YG_k}^*(V)$. In other words, this graph is sparser, which is also verified by Figure 5. Notice that theoretically, $YY_k(V)$ and ${YG_k}^*(V)$ have bounded node in-degree and out-degree. It can be shown from the maximum node degrees in Table 1.

	d_{avg}	d_{max}	I_{avg}	I_{max}	O_{avg}	O_{max}
UDG	23.57	48	-	-	-	-
RNG	2.38	5	-	-	-	-
GG	3.57	8	-	-	-	-
YG	9.05	20	6.67	18	6.67	8
YG^*	5.29	10	4.79	10	4.79	8
YY	5.02	9	4.65	8	4.65	8
YS	4.28	8	-	-	-	-

Table 1. Node degrees of different topologies.

4.2 Power Stretch Factor

Besides strong connectivity, the most important design metric of wireless networks is perhaps the energy efficiency, as it directly affects both the node and the network lifetime. So while our new topology increases the sparseness, how does it affect the power efficiency of the constructed network? Table 2 summarizes our experimental results of the length and power stretch factors of all the topologies. Here, t_{avg}/t_{max} is the average/maximum length stretch factor; ρ_{avg}/ρ_{max} is the average/maximum power stretch factor. Remember that we only prove that if $\alpha = \left(2\sin\frac{\pi}{k}\right)^{\beta} < \lambda^{\beta}$, *i.e.*, $2\sin\frac{\pi}{k} < \lambda$, $YY_k(V)$ has a bounded stretch factor. Thus, usually k is very large if λ is small. However, in our experiments we choose $k = 8$. It is not surprise that the average and the maximum node degree of the new topology $YY_k(V)$ are much smaller than those of $YG_k(V)$ and ${YG_k}^*(V)$. However, it is surprise that the average and the maximum power stretch factors of $YY_k(V)$ are at the same level of those of the $YG_k(V)$ and ${YG_k}^*(V)$.

5 Conclusion

In this paper, we propose a new spanner for wireless ad hoc networks, which can be constructed efficiently in a distributed manner. We prove that this new topology not only has a bounded node degree k but also has a bounded power stretch factor 2 in civilized graph. Until now, we always assume that the wireless devices are static or rarely move in a

	t_{avg}	t_{max}	ρ_{avg}	ρ_{max}
UDG	1.000	1.000	1.000	1.000
RNG	1.319	4.549	1.056	3.509
GG	1.124	1.991	1.000	1.000
YG	1.041	1.723	1.002	1.461
YG^*	1.070	1.895	1.003	1.461
YY	1.074	1.895	1.004	1.461
YS	1.090	2.174	1.004	1.473

Table 2. Length (Power) stretch factors.

reasonable time period. However, it is not difficult to update the graph $\overrightarrow{YY}_k(V)$ when the wireless nodes are moving because the existence of an edge $\overrightarrow{uv}$ only depends on some cone using u as apex, which contains v.

Even the graph $\overrightarrow{YY}_k(V)$ has a good power stretch factor in practice generally and a bounded power stretch factor in civilized graph. It is still an open problem whether it is a spanner theoretically in general graph.

In [9], we also consider another undirected structure, called *symmetric Yao graph* $YS_k(V)$, which guarantees that the node degree is at most k. It is constructed as follows. Each node u divides the region into k equal angular regions centered at the node, and chooses the closest node in each region, if any. An edge uv is selected to graph $YS_k(V)$ if and only if both directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are in $\overrightarrow{YG}_k(V)$. Then it is easy to show that the symmetric Yao graph is a subgraph of the Yao plus reverse Yao graph, and the relative neighborhood graph is a subgraph of the symmetric Yao graph. Therefore, the subgraphs of $UDG(V)$ mentioned in this paper have the following relations:

$$RNG(V) \subseteq YS_k(V) \subseteq \overrightarrow{YY}_k(V) \subseteq \overrightarrow{YG}^*_k(V) \subseteq \overrightarrow{YG}_k(V).$$

Our experiments also show that $YS_k(V)$ has a small power stretch factor in practice. Figure 6 shows an example of $YS_k(V)$. From the last rows of Table 1 and Table 2, we can see the average degree of $YS_k(V)$ is much smaller than that of $\overrightarrow{YY}_k(V)$, but its power stretch factor is still small enough. So an interesting open problem is whether $YS_k(V)$ has a bounded stretch factor.

Notice that, recently Li *et.al.* [8] introduced a similar structure. However, they also can not prove whether their structure has a bounded stretch factor.

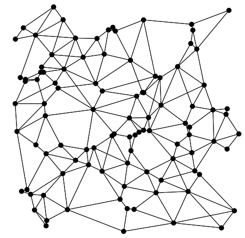

Figure 6. $YS_8(V)$ **generated from** $UDG(V)$**.**

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