Simple Heuristics and PTASs for Intersection Graphs in Wireless Ad Hoc Networks

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ABSTRACT

In wireless ad hoc networks, each wireless device has a transmission range, which is usually modeled as a disk centered at this node. A wireless node can send message directly to all nodes lying inside this disk. We present several intersection graphs to model the wireless networks. Then we present some simple heuristics and/or PTASs to approximate the maximum independent set, the minimum vertex cover and the minimum graph coloring in these graph models.

Categories and subject descriptors: C.2.1 Network Architecture and Design—Wireless communication; F.2.2 Nonnumerical Algorithms and Problems—Geometrical problems and computations; G.2.2 Graph Theory—Network problems.

General Terms: Algorithms.

keywords: Independent set, vertex cover, vertex coloring, disk graphs, PTAS, wireless ad hoc networks.

1. INTRODUCTION

Mobile wireless networking has received significant attention over the last few years due to its wide potential applications in various situations such as battlefield, emergency relief and so on. There are no wired infrastructures or cellular networks in wireless *ad hoc* networks. Multi-hop communication (carried out by the relaying of intermediate nodes) is required when the receiver node is not within the sender's transmission range. Thus, each wireless node also acts as a router, forwarding data packets for other nodes. Each wireless node often has an omni-directional antenna, i.e., a single transmission of a node can be received by any node within its vicinity. The transmission range of a node is often modeled as a disk centered at this node. When all wireless nodes have the same transmission radius, the wireless ad hoc networks are often modeled by the unit disk graph (UDG), in

which two nodes are connected iff their Euclidean distance is no more than the transmission radius. However, generally, different nodes will have different transmission radius due to various workloads and tasks performed. Thus, we apply the intersection graph and its variations to model the wireless networks.

Intersection graphs are graphs each of whose vertices is represented by a set, and two vertices are connected if their corresponding sets have a non-empty intersection. Intersection graphs have been widely studied [4, 11, 12]. The example of intersection graphs includes interval graph [9], unit disk graph [1, 2, 10], and coin graph [14]. In this paper, we define several new intersection graph models to capture the communication properties of wireless networks. It is well-known that many *NP-hard* problems remain NP-hard even they are restricted to intersection graph model. Moreover, the maximum independent set (MIS) problem, the minimum vertex cover (MVC) problem, and the minimum graph coloring (MGC) problem remain NP-hard even we restrict them to unit disk graph. In this paper we present some simple heuristics and PTASs to approximate these problems in the intersection graph models.

1.1 Preliminaries

We consider a wireless ad hoc network consisting of a set V of wireless nodes distributed in a two-dimensional plane. In addition, we assume that the nodes are static or can be viewed as static during a reasonable period of time. Each node $v \in V$ has a transmission radius r_v. Let $D(v, r_v)$ denote the disk centered at v with radius r_v. Then the set of wireless node V defines a set of disks $\mathcal{D} = \{D(v, r_v) \mid v \in V\}$ in the two dimensional plane. Generally, without specifically expressing, we assume that the geometry location of a node v is unknown, but the transmission radius r_v is known by node v and each node v knows all its 1-hop neighbors in the graph models (proposed later). Let $\|uv\|$ denote the Euclidean distance of two wireless nodes u and v.

A *disk graph* is the intersection graph of the set of disks defined by wireless nodes V. In other words, two nodes u and v are connected if $\|uv\| \leq r_u + r_v$. In wireless ad hoc networks, if two nodes u and v are not connected in the disk graph, then they can transmit messages simultaneously without causing any signal interferences. Therefore, the chromatic number of the disk graph is the upper-bound of the number of channels needed by all wireless nodes so that they can communicate without interferences. Here we

assume that a wireless node can tune its receiving device to different channels other than its transmission channel. The disk graph model actually captures the *primal interference* among all wireless nodes.

However, the disk graph model is an over-estimation for interferences. For example, when two disks intersect, i.e., the transmission region of two corresponding wireless nodes overlap, these two nodes can still use the same channel if their common transmission region does not contain any other wireless node inside. Then we define the *interference graph* as follows: two nodes u and v are connected if there is a node inside their common transmission region, i.e., $D(u, r_u) \cap D(v, r_v)$ contains some node from V. The interference graph model takes not only the primal interferences but also the secondary interferences into account.

Remember that, for wireless ad hoc networks, when node u is not within the transmission region of v, then v cannot directly communicate with u; it needs some other intermediate nodes to relay the message. When a node v sends a message to some node w within its transmission range, several existing wireless protocols require a direct acknowledgment from node w. Then we can model the ack-based communication by the *ack-communication graph*. Two nodes u and v are connected if $\|uv\| \leq \min(r_u, r_v)$.

The last interesting model is the *conflict graph*, in which two nodes u and v are connected, i.e., conflicted, if $\|uv\| \leq \max(r_u, r_v)$. Then the chromatic number of the conflict graph is a lower-bound of the number of channels needed by all wireless nodes so that they can communicate without interferences.

For a given set of wireless nodes V with a function R specifying the transmission radius of each node, we let $DG(V, R)$, $IG(V, R)$, $AG(V, R)$, and $CG(V, R)$ denote the disk graph, the interference graph, the ack-communication graph, and the conflict graph defined by V and R. When it is clear from the context, we will omit R or even V. It is easy to show that $AG \subseteq CG \subseteq IG \subseteq DG$.

A subset of vertices in a graph G is an *independent set* if for any pair of vertices, there is no edge between them. It is a *maximal independent set* if no more vertices can be added to it and it still be an independent set. It is a *maximum independent set* (MIS) if no other independent set has more vertices. The goal of the MIS problem is to compute, given a graph, a subset of pair-wisely unconnected vertices with maximum cardinality.

Given a graph $G = (V, E)$, a subset $V' \subseteq V$ of vertices is a *vertex cover* if for every edge in G, V' contains at least one of its end-vertices. In other words, every edge is dominated by some node from V'. The goal of *minimum vertex cover* (MVC) problem is to find a vertex cover with minimum cardinality. If the vertex of the graph is weighted, then the minimum weighted vertex cover problem is to find a vertex cover with minimum total weight.

The *graph coloring* (GC) problem (often called vertex coloring problem) is to assign each vertex a color so that two adjacent vertices have different colors. The *minimum graph coloring* (MGC) problem is to use minimum number of colors so a valid vertex coloring can be obtained.

In this paper, we are interested in designing efficient algorithms to approximate the maximum independent set, the minimum vertex cover, and the minimum graph coloring for the graph models defined above.

Given a graph G and a problem P, let $OPT_P(G)$ de-note an optimum solution of problem P when the input graph is G. An algorithm is a ρ-approximation algorithm for a problem P if, given any input graph G, it runs in polynomial time and always computes a solution that is at least $\frac{1}{\rho} OPT_P(G)$ for maximization-based P and at most $\rho \cdot OPT_P(G)$ for minimization-based P. An algorithm is a *polynomial-time-approximation-scheme* (PTAS) if, for any additional parameter $\varepsilon > 0$, it always computes a solution that is at least $\frac{1}{1+\varepsilon} OPT_P(G)$ for maximization based problem P and at most $(1+\varepsilon) \cdot OPT_P(G)$ for minimization based problem P.

1.2 Previous Results

For unit disk graphs, Marathe *et al.* [10] gave simple centralized heuristics to approximate the maximum independent set, the minimum vertex cover, the minimum vertex coloring, the minimum dominating set, and the minimum connected dominating set within constant 3, $\frac{3}{2}$, 3, 5, and 10 respectively. Hunt *et al.* [8] then presented the first PTAS to approximate the maximum independent set, the minimum vertex cover, and the minimum dominating set in UDG.

For disk graphs, it was claimed in [10] that the MIS problem can be approximated within 5. Then Erlebach [3] proposed an elegant PTAS for maximum weighted independent set (MWIS) and the minimum weighted vertex cover (MWVC) based on the shifting strategy used by Hochbaum [7] and Hunt *et al.* [8]. The algorithm runs in time $\frac{1}{\varepsilon^2} n^{O(\frac{1}{\varepsilon^4})}$. Thus, when the wireless network is modeled by a disk graph, we already have PTASs for maximum independent set and minimum vertex cover problems.

1.3 Our Results

In this paper, we present PTASs or efficient algorithms to approximate MIS, MVC and MGC when the wireless networks are modeled by disk graphs, interference graphs, ack-communication graphs, and conflict graphs. Some of our algorithms need only every node's transmission radius and the graph structure, while the PTASs need the exact geometry location of each node as extra. We present PTASs for MIS and MVC when the network is modeled by disk graphs, interference graphs, ack-communication graphs, and conflict graphs. We also show that the graph coloring problem in these graph models can be approximated with 5 for disk graphs, ack-communication graphs, and conflict graphs and 40 for interference graphs.

The rest of the paper is organized as follows. In Section 2, we show that the previous centralized method for computing MIS in disk graph model still works in other models introduced in this paper. However, this method does not extend naturally to minimum dominating set. We also give a PTAS for approximating the maximum weighted independent set. In Section 3, we present modified algorithms suitable for approximating the minimum weighted vertex cover. Section 4 is devoted to study the approximation of graph coloring. We conclude our paper in Section 5 by pointing out some possible future research directions.

For easy reading, we summarize the notations used in this paper in Table 1. In these notations, the definition of *in-dependence* and *non-independence* are different in various graph models. Figure 1 shows the situations in which two disks D_v and D_u are not independent under different graph models. Similarly, it may have different meaning that a disk D_i could *intersect* a geometry square S. In the disk graph

model, the conflict graph model, and the interference graph model, a disk D_i is said to *intersect* a geometry square S if it intersects the square S geometrically. However, in the ack-communication graph model, we apply a much stronger condition: a disk D_i is said to *intersect* a geometry square S if its center v_i is inside S. The reason is that if v_i is outside of S, then it is impossible for v_i to connect to any node v_j with the disk D_j contained inside S. Therefore, for ack-communication graph model, we only consider the disks whose centers are inside S instead of all the disks which geometrically intersect S in our algorithms. We say a disk D_i *covers* a disk D_j if v_iv_j is an edge in the corresponding graph model.

(a) ack-communication   (b) conflict graph

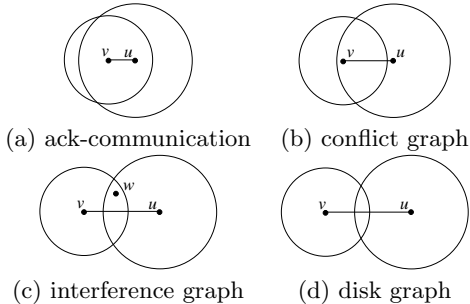(c) interference graph   (d) disk graph

Figure 1: Non-independence in our graph models.

2. MAXIMUM INDEPENDENT SET

The problem of constructing MIS is well-known to be a NP-hard problem even restricted to UDG [2]. It is easy to show that UDG is a special case of all graph models introduced in this paper. Thus, constructing MIS remains NP-hard when restricted to the graphs defined in this paper. For disk graphs, Marathe *et al.* [10] presented an algorithm that is easy to implement and approximates the MIS within 5. Recently, Erlebach [3] proposed a PTAS for MIS on disk graph based on the shifting strategy [7, 8]. In this section, we show that these methods work well in all graph models.

2.1 Simple Approximation Method

Assume that we know the transmission radius r_v of each wireless node v. In addition, we also know the graph representation of the underlying graph. The algorithm (see [10]) first finds the node with the smallest transmission radius, and adds it to the independent set; then removes this node and all its neighbors from the graph. Repeat the above steps until the graph is empty. Obviously, this algorithm does compute a maximal independent set with $O(n \log n)$ running time. We then prove the following theorem that guarantees the quality of the computed independent set.

THEOREM 1. *The computed IS has size at least $\frac{1}{5}$ of that of MIS if the input graph is a disk graph, an ack-communication graph, or a conflict graph.*

PROOF. We claim that, for any node $u \in IS$, there are at most 5 nodes from an optimum MIS solution such that they are connected to u and are removed by the algorithm due to the removing of u. Notice that, every node from an optimum solution is connected to some node from IS

computed by our algorithm because IS is maximal. Then it immediately implies our theorem.

To prove our claim, we will show that if v_1 and v_2 from an optimum solution are connected to u and removed by u, then $\angle v_1 u v_2 > \frac{\pi}{3}$. The selection of node u always implies that $r_u \leq r_{v_i}$ for $i = 1, 2$. We prove this for different input graph models.

If the input graph is a disk graph, uv_i is an edge implies that $\|uv_i\| \leq r_u + r_{v_i}$ for $i = 1, 2$. Notice that, here, nodes v_1 and v_2 cannot belong to IS. Because v_1 and v_2 are independent, $\|v_1v_2\| > r_{v_1} + r_{v_2}$. Thus, $\|v_1v_2\| > \|uv_i\|$ for $i = 1, 2$. In other words, $\angle v_1 u v_2 > \frac{\pi}{3}$.

If the graph is an ack-communication graph, uv_i is an edge implies that $\|uv_i\| \leq \min(r_u, r_{v_i}) = r_u$ for $i = 1, 2$. Because v_1 and v_2 are independent, $\|v_1v_2\| > \min(r_{v_1}, r_{v_2}) \geq r_u$. Thus, $\|v_1v_2\| > \|uv_i\|$ for $i = 1, 2$. In other words, $\angle v_1 u v_2 > \frac{\pi}{3}$.

If the graph is a conflict graph, uv_i is an edge implies that $\|uv_i\| \leq \max(r_u, r_{v_i}) = r_{v_i}$ for $i = 1, 2$. Because v_1 and v_2 are independent, $\|v_1v_2\| > \max(r_{v_1}, r_{v_2})$. Thus, $\|v_1v_2\| > \|uv_i\|$ for $i = 1, 2$. In other words, $\angle v_1 u v_2 > \frac{\pi}{3}$. □

Notice that the above proof cannot be directly applied to interference graphs. Unlike the proof of Theorem 1, we cannot show that, if v_1 and v_2 from an optimum solution are connected to u and removed by u, then $\angle v_1 u v_2 > \frac{\pi}{3}$. Instead, we have

THEOREM 2. *The computed IS has size at least $\frac{1}{40}$ of that of MIS if the input graph is an interference graph.*

PROOF. We prove this using an area argument. Consider any node u selected by our algorithm. The nodes from an optimum solution, which are connected to u and removed by u, can be partitioned into two cases: outside D_u or inside D_u.

Let $v_1, v_2, \cdots, v_k \notin D_u$ be the k nodes from the optimum solution that are connected to u and are removed by the algorithm due to the removing of u. The selection of u implies that $r_{v_i} \geq r_u$ for all $i = 1, 2, \cdots, k$. In addition, the disk D_i centered at v_i, $i = 1, 2, \cdots, k$ cannot contain any node v_j, $j \neq i$ inside because all disks centered at $i = 1, 2, \cdots, k$ are independent in the interference graph model. Nodes v_i, $i = 1, 2, \cdots, k$ is connected to u implies that D_i intersects with disk D_u. Let B_u be the disk centered at u with radius $2r_u$ and $T_u = B_u - D_u$. Then every disk D_i, $i = 1, 2, \cdots, k$, will intersect T_u. It is not difficult to show that $T_u \cap D_i$ achieves the smallest area when v_i is on the boundary of B_u and $r_{v_i} = r_u$. See Figure 2. We can show that $\angle wv_2u > \frac{5}{12}\pi$. Thus, the smallest area is at least $\frac{5}{12}\pi$. Notice that the area of T_u is 3π.

Notice that the region $T_u \cap D_i$ and $T_u \cap D_j$ for $1 \leq i, j \leq k$ may overlap. However, we will show that every point x is covered by at most 5 disks from D_i, $i = 1, 2, \cdots, k$. See Figure 3. Assume node x is covered by two disks D_i and D_j, i.e., $\|xv_i\| \leq r_{v_i}$ and $\|xv_j\| \leq r_{v_j}$. Then $\|v_iv_j\| > \max(r_{v_i}, r_{v_j})$ because D_i and D_j are independent in the interference graph, which implies that $\angle v_i x v_j > \frac{\pi}{3}$. Thus, x is covered by at most 5 independent disks. Therefore, by an area argument, we have $k \cdot \frac{5}{12}\pi < 5 \cdot 3\pi$. Thus, $k \leq 35$.

Let $v_1, v_2, \cdots, v_h \in D_u$ be the h nodes from the optimum solution that are connected to u and are removed by the algorithm due to the removing of u. Then obviously, all
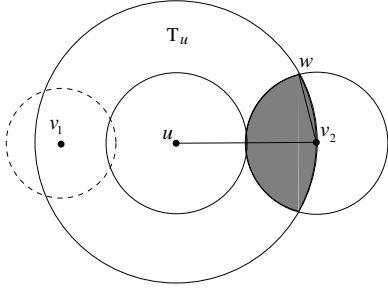
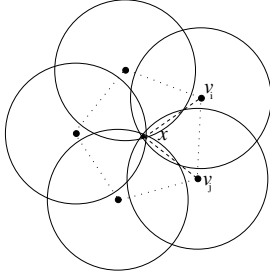**Figure 2: The number of independent neighbors removed by u is bounded.**

Figure 3: The number of independent disks which cover any point x is bounded by 5.

disks centered at v_i, $i = 1, 2, \cdots, h$ contain node u but not any other node. Then similarly, we have $\angle v_i u v_j > \frac{\pi}{6}$ for $1 \leq i, j, \leq h$. Thus $h \leq 5$.

Consequently, there are at most $35 + 5 = 40$ independent nodes are removed when we remove all nodes adjacent to a node u selected by our algorithm. □

Figure 4 gives a configuration example such that there are 12 independent nodes that are removed. These 12 nodes v_i, $i = 1, 2, \cdots, 12$ are equally distributed on the circle centered at u with radius $2r_u$. The transmission radius of each of them is set as $r_u + \epsilon$ for a very small positive ϵ. A node w_i, $i = 1, 2, \cdots, 12$, is placed on segment uv_i and $\|uw_i\| = r_u$. Thus, node u has interference with all 12 nodes v_i, $i = 1, 2, \cdots, 12$, and these 12 nodes are independent with each other. It then implies that the lower bound on the
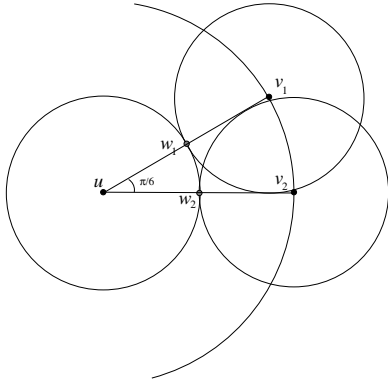

Figure 4: 12 independent neighbors removed by u.

approximation ratio for MIS of the smallest radius heuristic

is at least 12 for interference graph model. We believe that 12 is also the tight upper bound, but to prove that, a more rigorous analysis is needed.

This simple approximation method for MIS can be easily converted to a distributed one that is suitable wireless ad hoc networks. It is omitted here due to space limit.

2.2 PTAS for MWIS

Assume that we are given a set $\mathcal{D} = \{D_1, D_2, \cdots, D_n\}$ of n disks in the two-dimensional plane, where disk D_i has transmission radius r_i, center $v_i = (x_i, y_i)$ and a weight $w(D_i) = w_i$. For a subset of disks $U \subseteq \mathcal{D}$, let $w(U) = \sum_{D_i \in U} w(D_i)$, i.e., the summation of the weights of disks in U. Two disks D_i and D_j are said to be *independent* if the two nodes v_i and v_j are not connected in the corresponding graph model. Figure 1 shows the *non-independence* in different graph models.

The shifting strategy [8] was used to develop a PTAS for the unit disk graph. The plane is subdivided into grid of size at m by m for some integer m by a collection of vertical lines $x = i \cdot m$ and horizontal lines $y = j \cdot m$. A subdivision is called (r, s)-shifting if it is formed by a collection of vertical lines $x = i \cdot m + r$ and horizontal lines $y = j \cdot m + s$. A square is formed by two consecutive vertical lines and two consecutive horizontal lines in a (r, s)-shifting. For each square, an optimal solution of MIS is obtained in polynomial time for all disks contained in the square but not intersecting the boundary of the square. The union of the MIS in all squares is returned as the final solution for this shifting. It is not difficult to show that the result is an independent set. The pigeonhole principal implies that there is a shifting of the subdivision such that the size of the computed independent set is at least $(1 - \frac{1}{m})^2$ of optimum.

The basic approach of the PTAS for approximating MIS in various graph models introduced in this paper is similar to that of [3], i.e., to divide the transmission disks into different level according to its radius. At same level, all nodes have similar transmission radii. As in [3], we scale all disks so that the largest disk has transmission radius $\frac{1}{2}$. Let r_{min} be the smallest transmission radius among all wireless nodes. Let $k > 1$ be a fixed integer and $\ell = \left\lfloor \log_{k+1} \frac{1}{2r_{min}} \right\rfloor$. We partition $\mathcal{D}$ into $\ell + 1$ levels such that level j, $0 \leq j \leq \ell$, consists of all disks D_i with transmission radius satisfying

$$\frac{1}{(k+1)^{j+1}} < 2r_i \leq \frac{1}{(k+1)^j}.$$

Let $l(D_i)$ denote the level of disk D_i, i.e., $l(D_i) = \left\lfloor \log_{k+1} \frac{1}{2r_i} \right\rfloor$.

As did for UDG, for each level j, we subdivide the plane into grid by using a set of vertical lines $L_{j,v} : x = v \frac{1}{(k+1)^j}$, $v \in Z$ and a set of horizontal lines $H_{j,h} : y = h \frac{1}{(k+1)^j}$, $h \in Z$. Hereafter j is called the level of the lines; v (and h) is called the index of the vertical (and horizontal) line $l_{j,v}$ (and $h_{j,h}$) at level j. A (r, s)-shifting of the subdivision is the grid defined by the set of vertical lines whose indices modulo k equal r and the set of horizontal lines whose indices modulo k equal s. It was proved in [3] that a vertical line at level j of a (r, s)-shifting subdivision is also a vertical line at level $j + 1$ of the (r, s)-shifting subdivision.

Any two consecutive vertical lines at level j whose indices modulo k equal r, and any two consecutive horizontal lines at level j whose indices modulo k equal s, form a *j-square* in the (r, s)-shifting subdivision. See Figure 5 for an illus-

tration of a 0-square for $r = s = 0$ and $k = 3$. Here the solid lines are all lines at level 0 and all dashed lines are all lines at level 1. The j-squares are represented by thicker lines. It is easy to show that any j-square S is subdivided into $(k+1)^2$ $(j+1)$-squares (by lines $L_{j+1,v}$ and $H_{j+1,h}$ at level $j+1$). Notice that it only contains k^2 grids defined by lines at level j. These $(j+1)$-squares S' are called the *children* of S, denoted by $S' \prec S$. And S is called the *parent* of S'. It is easy to show that any j-square S has length $\frac{k}{(k+1)^j}$. Thus it can contain some disks inside with level at least $j-1$, but not any disks inside with level less than $j-1$.

A disk D_i with center (x_i, y_i) and radius r_i is said to hit a vertical line at $x = a$ if $a - r_i < x_i \le a + r_i$. Similarly, we say the disk D_i hits a horizontal line at $y = b$ if $b - r_i < y_i \le b + r_i$. In other words, a disk hits a line if it intersects this line, except that it only touches the line from the left or from the bottom. For maximum independent problem, a disk D_i at level $l(D_i) = j$ is said to be *active* (respecting to shifting (r, s)) if it does not intersect the boundary of any j-square of the (r, s)-shifting subdivision. The definition of active disks for minimum vertex cover is different, which will be discussed later. For a j-square S, let $\mathcal{D}_S^{<j}$ be the set of active disks with level less than j and intersect with S. Similarly, we define $\mathcal{D}_S^{\le j}$, $\mathcal{D}_S^{=j}$, $\mathcal{D}_S^{>j}$, and $\mathcal{D}_S^{\ge j}$. For a j-square S, let $OPT_S^{<j}$ be the set of disks from $OPT(\mathcal{D}_S)$ with level less than j and intersect with S. Similarly, we define $OPT_S^{\le j}$, $OPT_S^{=j}$, $OPT_S^{>j}$, and $OPT_S^{\ge j}$.

For each level j, let $\mathcal{D}_j(r, s)$ be the set of active disks at level j respecting to shifting (r, s). Define $\mathcal{D}(r, s) = \cup_{j=0}^\ell \mathcal{D}_j(r, s)$, i.e., the union of active disks at all levels respecting to shifting (r, s). Then a j-square S is called *relevant* if $\mathcal{D}(r, s)$ contains at least one disk of level j that is inside S in the corresponding graph model. More rigorous definition of *relevant* will be given later, which depends on the graph models. Let $OPT_{IS}(\mathcal{D}', G)$ denote the weight of the maximum weighted independent set for a set of disks $\mathcal{D}'$ when the network is modeled by graph model G. We will omit G and/or IS when it is clear from the context. The following proof is given by Erlebach [3]. We found that the correctness of this lemma does not depend on the graph model. We include it here for the completeness of presentation.

LEMMA 3. *There is at least one shifting (r, s), $0 \le r, s < k$ such that*

$$OPT(\mathcal{D}(r, s)) \ge (1 - \frac{1}{k})^2 \cdot OPT(\mathcal{D})$$

when the graph is modeled by the disk graph, the interference graph, the ack-communication graph, and the conflict graph.

PROOF. Consider a maximum weighted independent set $S^\star \subseteq \mathcal{D}$ for any graph model introduced here. Let $S_r^\star$ be the set of disks $D_i \in S^\star$ such that the disk D_i hits some vertical line $L_{j,v}$ at the level $j = l(D_i)$ whose index v modulo k equals r. Then $\cup_{r=0}^{k-1} S_r^\star \subseteq S^\star$. In addition, $S_r^\star$, $0 \le r \le k-1$ are pair-wise disjoint. Thus,

$$\sum_{r=0}^{k-1} w(S_r^\star) \le w(S^\star)$$

The pigeonhole principal implies that there is an index r_0 such that $w(S_{r_0}^\star) \le \frac{1}{k} \cdot w(S^\star)$. Let $S_{\overline{r_0}}^\star = S^\star - S_{r_0}^\star$. Therefore, $w(S_{\overline{r_0}}^\star) \ge (1 - \frac{1}{k}) \cdot w(S^\star)$.

Using the same technique, we can show that there is an index s_0 such that the set of disks from $S_{\overline{r_0}}^\star$, each D_i of which does not hit a horizontal line $H_{j,h}$ at its level $j = l(D_i)$ and the index h modulo k equals s_0, has total weight at least $(1 - \frac{1}{k}) \cdot w(S_{\overline{r_0}}^\star)$. Use $S_{\overline{r_0}, s_0}^\star$ to denote such set of disks. Obviously, $S_{\overline{r_0}, \overline{s_0}}^\star$ is an independent set for $\mathcal{D}(r_0, s_0)$. Thus, there is a (r_0, s_0)-shifting such that the weight of the maximum weighted independent set in $\mathcal{D}(r_0, s_0)$ is at least $(1 - \frac{1}{k})^2$ of the optimum $OPT(\mathcal{D})$. $\square$

Before we show the PTASs for maximum weighted independent set for graph models introduced in this paper, we first exam the structural properties of an optimum solution for all disks in $\mathcal{D}(r, s)$ for $0 \le r, s \le k - 1$. Given a graph model, an optimum solution cannot contain any disk that hits a line at level 0 of the (r, s)-shifting subdivision. In other words, each disk of the optimum solution $OPT(\mathcal{D}(r, s))$ is contained inside some 0-square. Moreover, the optimum solution can be divided into two subsets. One contains some independent disks at level 0, denoted by I_0. The other one contains independent disks at lower level that are independent with any disk from I_0. By the definition of $\mathcal{D}(s, r)$, all disks in the second subset cannot intersect any lines, with level 1, of the (r, s)-shifting subdivision. In other words, each disk in the second subset is contained inside some 1-square. Figure 5 gives an example of optimum solution in a 0-square. Here $k = 3$ and $r = s = 0$. The disks with the thickest boundary are at level 0. The disks that are not active is represented by dashed boundary. The shaded disks are in the optimum solution.
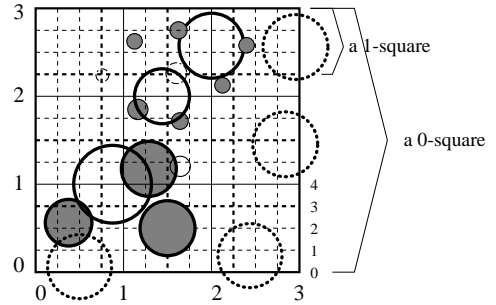

Figure 5: An optimum solution for a 0-square.

The above partition of disks in the optimum solution in a 0-square can be performed recursively down to the squares at level ℓ as follows. Given a j-square S, let I be a set of independent disks of level smaller than j, each of which intersects S. Let $MWIS(S, I)$ be a maximum weighted independent set of disks that are contained in S (must be of level at least j) and independent from the disks in I. Then the union of $MWIS(S, \emptyset)$ for all relevant squares S without parent must be the optimum solution for $\mathcal{D}(r, s)$.

We then discuss in detail how to compute $MWIS(S, I)$ using dynamic programming. Assume that we already compute the entry $MWIS(S, I)$ for all squares S with level at least $j + 1$ and all appropriate independent set I intersecting S. We show how to compute $MWIS(S, I)$ for a j-square S and some independent set of disks I intersecting S. The disks in $MWIS(S, I)$ can be divided into two subsets. One, denoted by X, contains some independent disks inside S with level j that are independent with disks from I. The other one contains independent disks at level larger than j

that are independent with any disk from I and X. By the definition of $\mathcal{D}(s,r)$, all disks in the second subset cannot intersect any lines, with level $j+1$, of the (r,s)-shifting subdivision. In other words, each disk in the second subset is contained inside some $(j+1)$-square S', which is contained in S. Thus, by properly choosing the set of disks X (disks inside S with level j and independent with I), we can generate $MWIS(S,I)$ by

$$MWIS(S,I) = \max_X \left(\left(\bigcup_{S' \prec S} MWIS(S', I_{S'} \cup X_{S'}) \right) \cup X \right).$$

Here $I_{S'}$ is the subset of disks from I that intersect S'. $X_{S'}$ is defined similarly.

It is not difficult to prove the correctness of the above dynamic programming approach. To guarantee that the above dynamic programming approach runs in polynomial time of the number of disks n and k, we have to show that the size of $I_S \cup X_S$, i.e., the number of independent disks with level at most j and intersecting a j-square S, is always bounded by a constant under the graph models introduced in this paper.

LEMMA 4 (BOUNDED INDEPENDENT DISKS). *Let S be any j-square and let I be a set of independent disks with level at most j, each of which may connect to some disk contained in S. Then there is a constant C depending on the graph model and k such that the cardinality of I is at most C.*

PROOF. We prove this lemma individually for each graph model introduced in this paper.

For disk graph model, Erlebach *et al.* [3] gave a constant $C = \frac{4}{\pi}(k+2)^2(k+1)^2$ using an area argument.
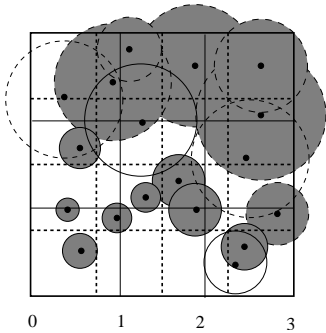

Figure 6: There are constant number of independent disks of level at most j, intersecting a j-square S in the ack-communication graph model.

For ack-communication graph model, the disk whose center is outside of S cannot have edge with any disk contained inside S. So if we add them to $MWIS(S,I)$, the resulted set is still guaranteed to be an independent set. Thus, unlike the disk-graph model, to bound $|I|$, we only have to consider the disks whose centers are inside S, i.e., the disks that can possibly "connect" with some disk contained inside S. Notice that all disks in I has level at most j, which implies that all disks in I has diameter at least $\frac{1}{(k+1)^{j+1}}$. Then the distance of the centers of any two disks D_p and D_q from I is at least $\frac{1}{2(k+1)^{j+1}}$; otherwise, these two disks D_p and

D_q will not be independent. The j-square S has side length $\frac{k}{(k+1)^j}$. Therefore, there are at most C_{AG} independent disks whose centers are inside S. See Figure 6. Here, by an area argument,

$$C_{AG} \le \left(\frac{k}{(k+1)^j} \right)^2 / \left(\pi \left(\frac{1}{4(k+1)^{j+1}} \right)^2 \right) = \frac{16k^2(k+1)^2}{\pi}.$$
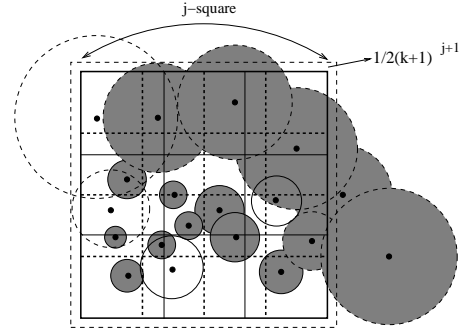

Figure 7: There are constant number of independent disks of level at most j and intersecting a j-square S in the conflict graph model.

For conflict graph model, again, we only have to consider all independent disks that could connect to some disks contained in S. Remember that, here, two disks D_p and D_q are independent if v_p is not inside D_q and v_q is not inside D_p. Thus, the distance between the centers of any two disks from I is at least $\frac{1}{2(k+1)^{j+1}}$. Using the same area argument, we can show that there are at most $\frac{16k^2(k+1)^2}{\pi}$ independent disks with centers inside S in the conflict graph model. Then we concentrate on estimating how many independent disks, denoted by I_O, such that (1) their centers are not inside S, (2) with level at most j, (3) each of them intersects S. We show that there are only a constant number of such disks by an area argument. Consider the four strips, denoted by $B(S)$, surrounded S with width $\frac{1}{2(k+1)^{j+1}}$. See Figure 7 for an illustration. For a disk $D_i \in I_O$, it is not difficult to show that $B(S) \cap D_i$ achieves the smallest area when v_i is on the boundary of $B(S)$ and $r_{v_i} = \frac{1}{2(k+1)^{j+1}}$. The smallest area of $B(S) \cap D_i$ is $\pi \frac{1}{8(k+1)^{2(j+1)}}$. Similar to Theorem 2, every point in $B(S)$ is covered by at most 5 disks from I_O. The area of $B(S)$ is $\frac{2k(k+1)+1}{(k+1)^{2(j+1)}}$. Thus, the size of I_O is at most

$$5 \cdot \frac{2k(k+1)+1}{(k+1)^{2(j+1)}} / \left(\pi \frac{1}{8(k+1)^{2(j+1)}} \right) = \frac{80k(k+1)+40}{\pi}$$

Thus, the total number of independent disks I with level at most j intersecting S is at most

$$C_{CG} \le \frac{16k^2(k+1)^2}{\pi} + \frac{80k(k+1)+40}{\pi}$$

For interference graphs, as we already showed that $CG \subseteq IG$, any independent set in IG is also an independent set in CG. We just proved that, for conflict graph model, the number of independent disks with level at most j intersecting S is bounded from above by constant C_{CG}. Thus, C_{CG} is also an upper-bound of the number of independent disks with level at most j intersecting S for interference graph model. $\square$

Remark: The number of independent disks that intersect a j-square in the ack-communication graph model is not bounded. It is easy to show that the disks D_i, $i \geq 1$, with center $v_i = (2^{i-1}(1+\epsilon)^i, 0)$ and radius $r_i = 2^{i-1}(1+\epsilon)^i$ are independent; all such disks intersect any square containing the point $(0,0)$. Here ϵ is a small positive real number.

The algorithm processes all relevant squares in order of non-increasing levels. For each j-square S and some appropriate independent set I, $MWIS(S,I)$ is computed by dynamic programming as follows.

ALGORITHM 1. *Compute MWIS(S, ∗)*

Let R be all disks in $\mathcal{D}(r,s)$ of level at most j intersecting S. For ack-communication graph model, we only consider disks of level at most j and with centers inside S.

For all $J \subseteq R$ with at most C disks {
  If J is an independent set then
    Let X be all disks in J with level j.
    For each child square S' of S {
      Let I' be disks in J intersecting S'.
      $X = X \cup MWIS(S', I')$. }
    Let I be disks in J with level less than j.
    If $w(X) > w(MWIS(S,I))$, then $MWIS(S,I) = X$. }

For the base situation, we can try all the possible independent sets to get the optimum solution in the largest level ℓ. The output of the algorithm is the union of the $MWIS(S, \emptyset)$, taken over all relevant squares S that do not have a parent. As did in [3], we can easily prove that the running time of this algorithm is $O(k^2 n^C)$. Here C is the constant in Lemma 4. In summary, we obtain a PTAS for the MWIS in all various intersection graphs introduced in this paper.

3. VERTEX COVER

It is well-known that the minimum weighted vertex cover for a general graph can be approximated within 2. In this section, we present a PTAS for the minimum weighted vertex cover problem for all graph models introduced here. As we did for approximating MWIS, we use the same partition of disks into levels and the same subdivision of the plane into squares at each level. In addition, all integer values r and s within interval $[0, k-1]$ are considered. We first study the structural properties of the MWVC. Let OPT be the minimum weighted vertex cover for $\mathcal{D}$. For any pair of r and s, Let $OPTV(r)$ be the set of disks $D_i \in OPT$ such that the disk D_i hits some vertical line $L_{j,v}$ at the level $j = l(D_i)$ whose index v modulo k equals r; Let $OPTH(s)$ be the set of disks $D_i \in OPT$ such that the disk D_i hits some horizontal line $H_{j,h}$ at the level $j = l(D_i)$ whose index h modulo k equals s. let $OPT(r,s) = OPTV(r) \cup OPTH(s)$.

LEMMA 5. *There is a (r_0, s_0)-shifting such that the total weight of disks in $OPT(r_0, s_0)$ is no more than $\frac{2}{k}$ of the weight of OPT.*

PROOF. Obviously, $OPTV(r)$ are pairwise disjoint for $0 \leq r < k$ and $\sum_{r=0}^{k-1} w(OPTV(r)) = w(OPT)$. Pigeonhole principal implies that there is r_0 such that $w(OPTV(r_0)) \leq \frac{1}{k} w(OPT)$. Similarly, there exists s_0 such that $w(OPTH(s_0)) \leq \frac{1}{k} w(OPT)$. Therefore, $w(OPT(r_0, s_0)) \leq w(OPTV(r_0)) + w(OPTH(s_0)) \leq \frac{2}{k} w(OPT)$. $\square$

Notice that the correctness of Lemma 5 does not depend on the graph model.

For independent set problem, to guarantee that the union of an independent set in each child of a square S is still an independent set, we only consider the independent disks that are totally contained in the children of S. In other words, a disk with level j is *active* for independent set if it is geometrically inside some j-square. Contrary to MWIS, a vertex cover of disks inside a square could use some disks are not totally inside in this square. Then we have to define active disks for a j-square differently. As the vertex cover has to cover all edges, we associate each edge to at least one square and then compute the vertex cover for all edges associated with that j-square.

For the disk graph model, the interference graph model, and the conflict graph model, we associate an edge uv to a j-square S if S intersects with $D_u \cap D_v$. Obviously, in any graph model introduced in this paper, if uv is an edge then D_u intersects D_v. In addition, there is always some square S that has non-empty intersection with $D_u \cap D_v$. Thus, any edge uv is associated with some square. See the left figure in Figure 8.

For the ack-communication graph model, we associate the edge uv to a j-square S if S intersects with $D_u \cap D_v$ and S contains u inside and either (1) $r_u < r_v$ or (2) $r_u = r_v$ and the identity $ID(u)$ of u is less than $ID(v)$. For any edge uv, assume that $u \in S_1$ and $r_u \leq r_v$. Then $S_1 \cap (D_u \cap D_v) \neq \emptyset$ because $u \in D_u \cap D_v$. So edge uv must be associated with square S_1. See the right figure in Figure 8.
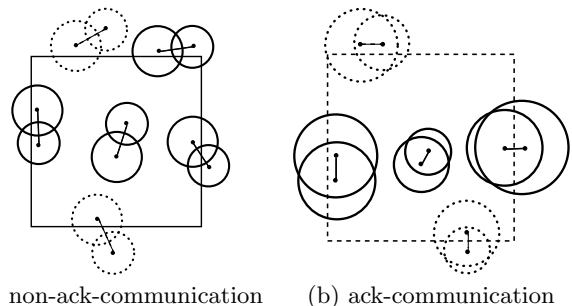

a) non-ack-communication   (b) ack-communication

Figure 8: Examples for the associated edges and the active disks in different graph models.

For a j-square S, a disk D_u is said to be *active* for S if there is an edge uv or vu associated with S for some v. Thus, for edges associated with a j-square S, their vertex cover must be a subset of the active disks in S. Notice that an active disk D_u for a j-square S always intersects S. See Figure 8 as an illustration. Here the dashed disks are inactive, the solid disks are active for S. For all graph models introduced here, each disk is at most active for four j-squares on the same level of this disk.

A j-square S is called *relevant* if there are some edges in the corresponding graph model associated with S. Given a j-square S, the presented algorithm will then construct a vertex cover for the edges associated with S using disks active for S knowing that some active disks P are already chosen to be in the vertex cover. More precisely, given some subset P of active disks for a j-square S, we will compute the minimum weighted set of active disks X, denoted by $VC(S, P)$, such that $P \cup X$ is a vertex cover for edges associated with S.

For each j-square S, we consider all the disks each of which is incident on some edges associated with S. We then consider the optimum solution OPT restricted to a square S, denoted by OPT_S, i.e., the disks in OPT that are *active* for S. Then, OPT_S is a vertex cover for all edges associated with S. Define *active edges* of S as the set of edges uv with D_u and D_v are active for S. Then the active edges of S contains the associated edges of S. Thus, OPT_S is also a vertex cover for all active edges of S.

Consider a relevant 0-square S, the optimum solution OPT_S can be divided into two subsets. One contains the disks which is a complement of an independent set of active disks for S at level 0, denoted by P_0. The other one contains disks at lower level that, together with P_0, can form a vertex cover for all edges active for S. Figure 9 gives an example of optimum solution restricted in a 0-square. Here $k = 3$ and $r = s = 0$. The disks with the thickest boundary are at level 0. The disks that are not active is represented by dashed boundary. The un-shaded disks are in the optimum solution. The shaded disks are an independent set.
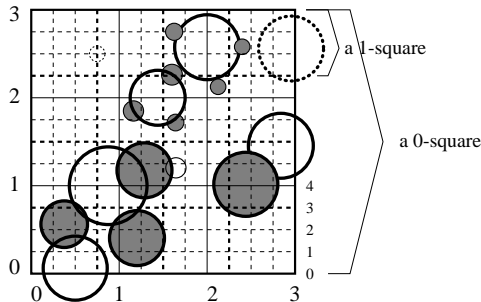

Figure 9: Active disks and a vertex cover for a 0-square in disk graph model.

The above partition of disks in the optimum solution in a relevant 0-square can be performed recursively down to the squares at level ℓ as follows. Given a j-square S, let R be the set of all *active* disks for S with level at most j. Then any *active* edge using only disks from R must have vertex cover from R. Thus for any vertex cover U, $R - U$ must be an independent set. Hereafter, we will use J to denote a set of *independent* and *active* disks with level at most j for a j-square S. Let $Q = R - J$ be the complement set of disks of J. Then, Q can cover all disk in R. Let $P = Q^{<j}$. Let $VC(S, P)$ be the set of minimum weighted disks such that $VC(S, P) \cup P$ is a vertex cover for all *active* edges for S. Notice that, for optimum solution OPT restricted to S, we can set $Q = OPT_S^{\leq j}$, $P = OPT_S^{<j}$, $J = R - OPT_S^{\leq j}$. The union of $VC(S, \emptyset)$ for all relevant squares S without parent must be a vertex cover for the graph defined on $\mathcal{D}$.

A dynamic programming to compute $VC(S, \emptyset)$ is straightforward. The algorithm presented here is similar to that of [3]. It processes all relevant squares in order of non-increasing levels. For each j-square S and some set P, $VC(S, P)$ is computed as follows.

ALGORITHM 2. *Compute* $VC(S, P)$

Let R be all *active* disks for S with level $\leq j$.
For all $J \subseteq R$ with at most C_2 disks {
  If J is an independent set then
    Let $Q = R - J$.

Let X be all disks in Q with level j.
For each relevant child square S' of S {
  Let P' be disks in Q that is active for S'.
  $X = X \cup VC(S', P')$. }
Let P be disks in Q with level less than j.
If $w(X) > w(VC(S, P))$, then $VC(S, P) = X$. }

For the base situation (without relevant child), we actually try all the complement sets of possible independent active disks to get the optimum solution with minimum weight. The output of the algorithm is the union of the $VC(S, \emptyset)$, taken over all relevant squares S that do not have a parent. Similar to Algorithm 1, the running time of this algorithm is $O(k^2 n^{C_2})$. Here C_2 is a constant that will be proved by the following Lemma 6.

LEMMA 6   (BOUNDED INDEPENDENT ACTIVE DISKS). *Let S be any j-square and let J be a set of independent disks with level at most j that is active for S. Then there is a constant C_2 depending on the graph model and k such that the cardinality of J is at most C_2.*

PROOF. For the disk graph models, the interference graph models, and the conflict graph models, all active disks intersect S. Then, the number of disks in J is bounded by a constant using an area argument, which is the same as Lemma 4.

We then consider the ack-communication graph model. For each disk $D_u \in J$ with center u outside S, there is a disk D_v with level at least j and uv is an edge. For ack-communication model, we know that u must be inside D_v. Thus, u is always inside the extended square S'. Similar to Lemma 4, the number of independent active disks is at most

$$C_2^{AG} \leq (\frac{1}{(k+1)^{j-1}})^2 / (\pi (\frac{1}{4(k+1)^{j+1}})^2) = \frac{16(k+1)^4}{\pi}.$$

This finishes the proof. $\square$

We then show that, given S and P, our algorithm does compute $X = VC(S, P)$ such that $X \cup P$ is a vertex cover for all edges associated with S. For any edge uv associated to S, without loss of generality, we assume that $r_u \leq r_v$.

If D_u has level j, then both D_u and D_v is in R. Obviously, not both of D_u and D_v can belong to some independent set J. In other words, either D_u or D_v or both is selected to Q. If D_u is selected to Q. Then D_u is also selected to X. So $X \cup P$ covers edge uv. If D_u is not selected to Q. Then D_v is selected to Q. Then D_v will be put into X if $l(D_v) = j$ or it will be put to P if $l(D_v) < j$. Thus, $X \cup P$ covers edge uv.

If D_u has level larger than j, then edge uv is also associated with some child S' of S. By induction, $VC(S', P')$ covers edge uv. From $VC(S, P) = X \cup \bigcup_{S' \prec S} VC(S', P')$, we know that uv is also covered by $VC(S, P)$.

Notice that every edge is active for some square. So the output of our algorithm is a vertex cover of $\mathcal{D}$. We then study the quality of the computed vertex cover. Let OPT be the minimum weighted vertex cover of $\mathcal{D}$ in a graph model introduced in this paper. The following theorem proved that the shifting strategy does work for MWVC problem. Same theory was proved in [3] for disk graph model.

THEOREM 7. *For all graph models introduced here, there is a (r,s)-shifting, $0 \le r, s < k$ such that for the vertex cover A generated by the algorithm,*

$$w(A) \le (1 + \frac{6}{k})w(OPT).$$

PROOF. Lemma 5 implies that there is a (r_0, s_0)-shifting such that $w(OPT(r_0, s_0)) \le \frac{2}{k}w(OPT)$. We then consider the subdivision of the plane that results from this choice of r_0 and s_0. Let $\mathcal{R}$ be the set of all relevant squares, and $\mathcal{R}_0$ be the set of all relevant squares without parent. For any j-square S, let $OPT_S^{=j}$ denote the disks with level j in OPT that are active for S. Notice that $OPT = \bigcup_{S \in \mathcal{R}} OPT_S^{=j}$. Any disk can be *active* for at most 4 squares on its level. In addition, if a disk is active for some square S, it must intersect S. Thus, when we summarize $w(OPT_S^{=j})$ over all relevant squares $S \in \mathcal{R}$, only disks in $OPT(r_0, s_0)$ are counted multiple times, whose weight $w(OPT(r_0, s_0)) \le \frac{2}{k}w(OPT)$. Since each such disk is counted at most four times, and it must be counted once in OPT, we have

$$\sum_{S \in \mathcal{R}} w(OPT_S^{=j}) \le w(OPT) + 3\frac{2}{k}w(OPT) = (1 + \frac{6}{k})w(OPT).$$

Then, we only need to prove that the total weight of the output $A = \bigcup_{S \in \mathcal{R}_0} VC(S, \emptyset)$ is at most $\sum_{S \in \mathcal{R}} w(OPT_S^{=j})$.

We already showed that the optimum solution OPT restricted to S, denoted by OPT_S, is also considered by the algorithm because in one iteration we have $Q = OPT_S^{\le j}$ and then $P = OPT_S^{<j}$. The algorithm computes the minimum weighted X such that $P \cup X$ is a vertex cover for all edges associated with S. Obviously,

$$w(VC(S, OPT_S^{<j})) \le \min_X w(X) + \sum_{S' \prec S} w(VC(S', P')),$$

where the minimum is taken over all X (a subset of active disks for S with level j) such that $R - X \cup OPT_S^{<j}$ is an independent set, and P' is all disks in $X \cup OPT_S^{<j}$ that are active for S'. Remember that here R is the set of all active disks with level at most j. Here, $OPT_S^{<j}$ denotes the set of all disks in OPT that have level $< j$ and are active for S; $S' \prec S$ means that S' is a child of S. We say a relevant j'-square $S'' \sqsubseteq S$ if $S'' = S$ or there exists a relevant square $S' \prec S$ and $S'' \sqsubseteq S'$. As did in [3], we will prove

$$w(VC(S, OPT_S^{<j})) \le \sum_{S' \sqsubseteq S} w(OPT_{S'}^{=j'}).$$

We prove it by induction on the number of relevant squares that were processed by the algorithm.

$$w(VC(S, OPT_S^{<j}))$$
$$\le \min_X w(X) + \sum_{S' \prec S} w(VC(S', P'))$$
$$\le w(OPT_S^{=j}) + \sum_{S' \prec S} w(VC(S', OPT_{S'}^{\le j}))$$
$$\le w(OPT_S^{=j}) + \sum_{S' \prec S} \sum_{S'' \sqsubseteq S'} w(OPT_{S''}^{=j''})$$
$$= \sum_{S' \sqsubseteq S} w(OPT_{S'}^{=j'}).$$

The second inequality is from taking $X = OPT_S^{=j}$. Thus, P' is all disks in OPT that are active for S' with level less than j'. Notice that, for any child S' of S, $OPT_{S'}^{\le j} = OPT_{S'}^{<j'}$. The third inequality is from induction. Therefore,

$$w(A) = \sum_{S \in \mathcal{R}_0} w(VC(S, \emptyset))$$
$$\le \sum_{S \in \mathcal{R}} w(OPT_S^{=j})$$
$$\le (1 + \frac{6}{k})w(OPT).$$

This finishes the proof of the theorem. □

In summary, we obtain a PTAS for the MWVC in all various intersection graphs introduced in this paper.

4. GRAPH COLORING

In this section, we study how to color the graphs introduced in this paper. Graf *et al.* [5] discussed four classes of disk graphs and studied the relations between their chromatic number and the clique number. They proved that, for all such four classes of graphs, their chromatic number is within a constant factor of the clique number. They considered the unit disk graphs, intersection disk graphs (corresponding to disk graph models here), containment disk graphs (corresponding to conflict graph models here), and double disk graphs. Here double disk graph is defined over a set of nodes. Each node u defines two disks $D(u, r_u)$ and $D(u, R_u)$. The double disk graph has an edge uv iff $D(u, r_u)$ intersects $D(v, R_v)$ or $D(v, r_v)$ intersects $D(u, R_u)$.

Let $\delta(G)$ denote the largest d such that G contains a subgraph H in which each vertex has degree at least d. It was proved by Szekeres and Wiff [13] that, every graph G can be colored in $\delta(G) + 1$ colors. Then Hochbaum [6] presented a method to find the value of $\delta(G)$ and gave an efficient method to color G using $\delta(G) + 1$ colors with only $O(|V| + |E|)$ time. For the completeness of presentation, we review the algorithm here. To evaluate $\delta(G)$, it dismantles G by successive removals of vertices of minimum degree and all incident edges. Let v_i denote the ith vertex removed from G and G_{i+1} be the graph after v_i is removed (set $G_1 = G$). The degree of v_i in graph G_i is called its valid degree. Set $\delta(G)$ as the maximum valid degree of all nodes. Let v_j be the node with the maximum valid degree $\delta(G)$. Then, v_j has $\delta(G)$ neighbors among the vertices $v_{j+1}, v_{j+2}, \cdots, v_n$. In, addition, all nodes v_i with $i > j$ have degree at least $\delta(G)$ in graph G_j. To color G in no more than $\delta(G) + 1$ colors, it scans the sequences of v_i's from v_n to v_1 and assigns to each v_i the smallest positive integer not yet assigned to any of its neighbors. Marathe *et al.* [10] also used this method to color the disk graphs and show an approximation factor of 6.

THEOREM 8. *The above coloring method achieves a constant approximation ratio for all graph models introduced in this paper.*

PROOF. It was already known that, for any graph G, G can be colored by $\delta(G) + 1$ colors. Let H be a subgraph such that all nodes have at least $\delta(G)$ degree in H. Let u be the node of H with the smallest radius. Let $N_H(u)$ be all neighbors of u in H. Then $|N(H)| \ge \delta(G)$.

Consider the induced coloring on $N(u)$ by any coloring of G. The node in $N(u)$ with the same color form an independent set. It was proved in Theorem 1 and Theorem 2 that, if u has a radius less than all its neighbors $N_H(u)$, then the maximum independent set in $N_H(u)$ has size at most 5, 5, 5 and 40 for the disk graph model, ack-communication graph model, conflict-graph model, and the interference graph model respectively. Let OPT be the chromatic number of the corresponding graph. There are only $OPT - 1$ colors for nodes in $N(H)$. Then $|N(H)| \leq 5(OPT - 1)$ for disk-graph, ack-communication graph and the conflict-graph, while $|N(H)| \leq 40(OPT - 1)$ for the interference graph. Thus, the colors used by the above method is at most $5 \cdot OPT - 4$ for disk-graph, ack-communication graph and the conflict-graph, while at most $40 \cdot OPT - 39$ for the interference graph. □

5. SUMMARY AND FUTURE WORK

In this paper, we present efficient algorithms and PTASs to approximate MIS, MVC and MGC problems when the wireless networks are modeled by disk graphs, interference graphs, ack-communication graphs, and conflict graphs. We show that MIS and MGC problems can be approximated within 5 or 40 for all graph models presented in this paper by simple methods. We also presents two PTASs for the weighted versions of MIS and MVC for all graph models.

One of the most challenging problems is to design a PTAS for dominating set problem and connected dominating set problem, if it is possible, when the wireless networks are modeled by some graph models introduced in this paper. This has been studied for unit disk graph, however, little is known for the graph models presented in this paper. We leave these problems as future work.

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Table 1: Notations used in this paper.

$\mathcal{D}$	a set of 2-dimensional disks
D_i	a disk with index i
$l(D_i)$	the level of a disk D_i
v_i	the center of a disk D_i
r_i	the transmission radius of a disk D_i
k	parameter to control the quality of solution
$\ell + 1$	the total levels of disks
$\mathcal{D}(r, s)$	disks that are *active* for shifting (r, s)
$MWIS(S, I)$	the maximum weighted set of *independent* disks inside a square S and are *independent* with disks in I
$VC(S, P)$	the minimum weighted set of disks such that $VC(S, P) \cup P$ *covers* all disks (with level at least j) inside a j-square S
U_S	the set of disks in U that are active for a j-square S
$U_S^{<j}$	the set of disks in U with level less than j that are active for S. Similarly we define $U_S^{\leq j}$, $U_S^{=j}$, $U_S^{>j}$, and $U_S^{\geq j}$.
R	the set of active disks with level at most j, intersecting a j-square S
J	a subset of *independent* active disks with level at most j, intersecting a j-square S
I	the set of *independent* active disks in J with level less than j that intersect a j-square S
Q	the set of active disks not in J, with level at most j, intersecting a j-square S. So $Q = R - J$.
P	the set of active disks not in J, with level less than j, intersecting a j-square S
C, C_2	upper-bounds on the number of independent active disks with level at most j that intersect a j-square