Capacity Bounds for Large Scale Wireless Ad Hoc Networks Under Gaussian Channel model

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***Abstract*—We study the *capacity* for both random and arbitrary wireless networks under *Gaussian Channel* model when all wireless nodes have the same constant transmission power P. During the transmission, the power decays along path with attenuation exponent $\beta > 2$. We consider extended networks, where n wireless nodes $\{v_1, v_2, \cdots, v_n\}$ are randomly or arbitrarily distributed in a square region B_a with side-length a. We randomly choose n_s multicast sessions. For each source node v_i, we randomly select k points $p_{i,j}$ $(1 \leq j \leq k)$ in B_a and the node which is closest to $p_{i,j}$ will serve as a destination node of v_i. We derive the achievable upper bounds on unicast capacity and an upper bound (partially achievable) on multicast capacity of the wireless networks under Gaussian Channel model. We found that the unicast (multicast) capacity for wireless networks under Gaussian Channel model has three regimes.**

***Index Terms*—Wireless networks, capacity, unicast, multicast, Gaussian channel.**

I. Introduction

Recently, the network capacity has been studied extensively under different network and system models, and different interference models. The ground breaking work of Gupta *et al.*, [3] has shown that when n wireless nodes are randomly placed in a square region with side-length 1, for randomly picked n pairs of source/destination nodes, the information exchangeable by each pair per unit time will go to zero in an order of $\frac{1}{\sqrt{n \log n}}$ as n tends to ∞ under some interference models, *e.g.*, protocol interference model (PrIM) and physical interference model. They also showed in [3] that even all nodes are located optimally, the amount of information that can be exchanged by each source/destination pair still goes to zero in an order of $\frac{1}{\sqrt{n}}$. In addition, the authors of [1], [2] proposed alternative technologies that achieve unicast capacity bound $\frac{1}{\sqrt{n \log n}}$ as in [3] for random wireless networks. Recently, Francheschetti *et al.* [4] proved that per-flow unicast capacity of order $\frac{1}{\sqrt{n}}$ is also achievable in networks of randomly located nodes when Gaussian channel model is used. Their scheme is based on *Percolation Theory*. They first construct a "highway" system (or say backbone) of the random wireless network. Then based on multihop transmission, pairwise coding and decoding at each hop, and a time division multiple access (TDMA) scheduling, they proved that the lower bound of unicast capacity for random wireless networks is $\frac{1}{\sqrt{n}}$. Hence, the unicast capacity gap between randomly and arbitrary wireless networks is claimed to be closed.

However, the work in [4] is based on Gaussian channel model, while the work in [3] is based on PrIM and physical interference models, and the results by applying the same scheme to different communication and interference models may be different. The main purpose of this paper is to study the *unicast capacity* (or more generally the multicast capacity) of large scale random or arbitrary wireless networks under Gaussian channel model when we choose the *best* protocols for all layers. Under Gaussian channel model, the data rate between any pair of transceivers (u, v) is determined by several parameters, including transmission power P of the transmitter u, the environment noise N_0 that can be heard by v, the interference signals from all other simultaneously transmitting nodes rather than u. Hence, multiple pairs of nodes can communicate directly with different data rates. For presentation simplicity, we assume that there is only one channel in a wireless network. And as always, we assume that data are sent from node to node either by one-hop or by multi-hop manner until they reach the destination. In addition, we assume every node has large enough buffer to save the relay traffic temporarily while waiting for being transmitted such that no packet will be lost through relaying.

Our Main Contributions: In this paper we derive analytical upper bounds and lower bounds of unicast(multicast) capacity for wireless networks under Gaussian channel model. We studied unicast(multicast) capacity for wireless networks with n wireless nodes (randomly or arbitrarily) distributed in a square region with side-length a . We studied different cases when a is in different ranges, *i.e.*, a is a function of n. Our main results are as follows.

For an arbitrary network in which we can optimally choose the locations of all nodes and choose pairs of communication nodes, the total unicast capacity $\Lambda(n)$ is:

$$\Lambda(n) = \begin{cases} \Theta(1) & \text{if } a = O(1) \\ \Theta(a^2) & \text{if } a = O(\sqrt{n}) \\ \Theta(n) & \text{if } a = \Omega(\sqrt{n}) \end{cases} \tag{1}$$

For random networks, the per-flow unicast capacity is

$$\varphi(n) = \begin{cases} \Theta(\frac{1}{n}) & \text{if } a = O(1) \\ \Theta(\frac{a}{n}) & \text{if } a = O(\sqrt{n}) \\ \Theta((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}}) & \text{if } a = \Omega(\sqrt{n}) \end{cases} \tag{2}$$

For a random wireless network, an upper bound (partially achievable) of minimum per-flow multicast capacity where

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each multicast flow will have k randomly chosen receivers, when $n_s = \Theta(n)$, is:

$$\varphi_k(n) = \begin{cases} O(\frac{a}{n\sqrt{k}}) & \text{if } a = O(\sqrt{n}) \\ \Theta(\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{k}}) & \text{if } a = \Theta(\sqrt{n}), k = O(\frac{n}{\log^2 n}) \\ O((\frac{a}{\sqrt{n}})^{-\beta} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{k}}) & \text{if } a = \Omega(\sqrt{n}), k = O(\frac{n}{\log^2 n}) \end{cases} \quad (3)$$

Consider a random wireless network, where nodes following a Poisson distribution with parameter $\frac{n}{a^2}$ are distributed in B_a (a square with side-length $a = \Omega(\sqrt{n})$). Assume that n_s random multicast flows are generated. Under Gaussian channel model, the per-flow multicast capacity $\varphi_k(n)$ is at most

$$\varphi_k(n) = \begin{cases} O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{n_s} \cdot \frac{\sqrt{n}}{\sqrt{k}}) & \text{if } k \leq \frac{n}{(\log n)^\beta} \\ O(\frac{n}{n_s k} (\frac{a}{\sqrt{n}})^{-\beta} (\log n)^{-\frac{\beta}{2}}), & \text{if } k \geq \frac{n}{(\log n)^\beta} \end{cases} \quad (4)$$

In contrast to [3], [4], studying unicast capacity of wireless networks under Gaussian channel model needs new technical insight. One of reasons is that the interference concept under Gaussian channel model is different from PrIM. Under PrIM, every node has fixed transmission range and interference range, the data rate between them is fixed as well. Compared with PrIM, the data rate under Gaussian channel model is determined by power, distance and noise. Any two nodes can communicate with each other although the data rate maybe go to zero when the distance between the transceiver pair is long or there are too much noise. Hence, some techniques used in previous work cannot be applied directly to Gaussian channel model without modification.

The rest of the paper is organized as follows. In Section II we discuss in detail the network model used in this paper. We present both upper-bounds and lower-bounds of unicast capacity for an arbitrary wireless network in Section III. The unicast capacity bounds for random wireless networks are presented in Section IV. We study the multicast capacity in Section V. We review the related results on network capacities in Section VI and conclude the paper in Section VII.

II. Network Model

Consider a square region B_a with side length a. We assume that there is a set $V = \{v_1, v_2, \cdots, v_n, \cdots\}$ of *ordinary* wireless terminals deployed in B_a following Poisson distribution with parameter $\frac{n}{a^2}$. In other words, given a region X with area x, the probability that there are exactly k nodes inside X is $\frac{(n/a^2)^k e^{-xn/a^2}}{k!}$. The expected number of nodes located in the region B_a is n and it is easy to show that, with high probability, the number of nodes is in the range $[(1-\epsilon)n, (1+\epsilon)n]$ for a small constant $0 < \epsilon < 1$. Thus, without affecting the asymptotic results, we assume that n is also the total number of nodes deployed. We randomly pick n_s wireless terminals as source nodes. Here, n_s can be as large as n such that every node will serve as a source node. For each source node v_i, we randomly select a point p_i in B_a and the node which is closest to p_i will become the destination node of v_i for unicast. Here, if the source node v_i chooses itself as its destination node, we can randomly generate point p_i again to avoid this. For studying multicast capacity, we assume that each multicast session will have k receivers. For each source node v_i, we randomly pick k points $p_{i,j}$, $1 \leq j \leq k$, in B_a and then the closest node $v_{i,j}$ to $p_{i,j}$ will serve as a destination node of the ith flow that has the source node v_i.

We assume that all nodes have a constant transmission power P, and for each transceiver pair (v_i, v_j), node v_j receives the transmitted signal from node v_i with power $P \cdot \ell(d(v_i, v_j))$, where $d(v_i, v_j)$ is the Euclidean distance between node v_i and v_j, $\ell(d)$ is the transmission loss during a path with length d. Here we consider the attenuation function

$$\ell(d) = \min\{1, d^{-\beta}\}$$

where the constant $\beta > 2$. Hence, any two nodes can establish a direct communication link over a unit bandwidth channel, of rate $R(v_i, v_j) = \log(1 + \frac{S(v_i,v_j)}{N_0 + I(v_i,v_j)}) = \log(1 + \frac{P \cdot \ell(v_i,v_j)}{N_0 + \sum_{q \neq i} P \cdot \ell(v_q,v_j)})$. Here, v_q is any other node which is transmitting simultaneously with v_i and N_0 is the variance of background noise, usually a constant, $I(v_i, v_j)$ is the total interference at the receiving node v_j when v_i and v_j communicate, and $S(v_i, v_j)$ denotes the strength of signal received by v_j sent from v_i.

We assume that any node v_i could serve as the source node for some unicast or multicast, here $1 \leq i \leq n$. And for each source node v_i, assume that node v_i will send data to its receiver(s) by unicast (or multicast) with a data rate λ_i. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n)$ be the *rate vector* of the multicast data rate of all multicast sessions. The total throughput of a feasible rate vector for unicast (multicast) is defined as $\Lambda_k(n) = \sum_{i=1}^n \lambda_i$. The average per flow unicast(multicast) throughput is defined as $\varphi_k(n) = \frac{\sum_{i=1}^n \lambda_i}{n_s}$, where n_s is the number of unicast (multicast) sessions, and k is the total number of nodes in each unicast(multicast) session, including the source node. Similarly, given n_s unicast(multicast) sessions with S as source nodes, the minimum per-flow multicast capacity is defined as

$$\varphi_k(n) = \min_{v_i \in S} \lambda_i.$$

In this paper, we will focus on the minimum per-flow capacity.

A. Useful Known Results

Throughout this paper, we will repeatedly use the following results from probability theory literature.

Lemma 1 (Azuma's Inequality): Suppose that random variables X_0, X_1, X_2, $\cdots$, X_n, $\cdots$ are martingale and $|X_k - X_{k-1}| \leq a_k$ almost surely for any $k \geq 1$. Then for all positive integers N and all positive real number t, we have

$$\mathbf{Pr}\left(|X_N - X_0| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^N a_k^2}\right)$$

A sequence of random variables X_i, $0 \leq i$, are called *martingale* if $\forall N > 0$, $E(X_{N+1} \mid X_0, X_1, \cdots, X_N) = X_N$. Here $E(X \mid Y)$ is the expected value of variable X with Y being true.

Lemma 2: [4] For a Poisson random variable X of parameter λ, $\mathbf{Pr}\left(X \geq x\right) \leq \frac{e^{-\lambda}(e\lambda)^x}{x^x}$, for $x > \lambda$.

NOTATIONS: Throughput this paper, for a continuous region B, we use $|B|$ to denote its area; for a discrete set S, we use $|S|$ to denote its cardinality; for a tree T, we use $\|T\|$ to denote its total Euclidean edge lengths; $x \to \infty$ denotes that variable x takes value to infinity.

B. Technical Lemmas

To study the asymptotic capacity, we first present several technical lemmas that are essential for the analysis. For a random wireless network with n wireless nodes located in a square region B_a, we partition B_a into cells with side length c. Two nodes u and v are said to have *cell-distance* d if they are located in two cells that are separated by d-cells.

Lemma 3: Based on a TDMA schedule, for any transceiver pair (u, v) with cell-distance d, the data rate $R(u, v)$ only depends on d and c. Furthermore, when $c \cdot d \to \infty$, $R(u, v) = \Omega(c^{-\beta} d^{-\beta-2})$.

We use the similar idea as Theorem 3 in [4] to prove this. See our report [5] for proof details.

Lemma 4: If we partition the square B_a into $\frac{a^2}{c^2}$ cells with constant side length c, then *w.h.p.*, there are less than $\frac{nc^2}{a^2} \log \frac{a}{c}$ nodes in each cell.

Proof: Let A_n be the event that there is at least one cell with more than $\log \frac{a}{c} \times \frac{nc^2}{a^2}$ nodes. Since the number of nodes x in each cell of the partition is a Poisson random variable of parameter $\frac{nc^2}{a^2}$, by the union the Chernoff bounds, we have $\Pr(A_n) \leq (\frac{a}{c})^2 \Pr(x > \log \frac{a}{c} \times \frac{nc^2}{a^2}) \leq (\frac{a}{c})^2 e^{-\frac{nc^2}{a^2}} (\frac{\frac{nc^2}{a^2} e}{\log \frac{a}{c} \times \frac{nc^2}{a^2}})^{\frac{nc^2}{a^2} \log \frac{a}{c}}$, which goes to 0 as $n \to 0$. ■

Lemma 5: If we partition square B_a into stripes with width a and height c_1, then with probability at least $1 - \frac{\sqrt{n}}{c_1} e^{-c_1\sqrt{n}} (\frac{e}{2})^{2c_1\sqrt{n}}$ the number of nodes in each stripe will be no more than $2\frac{c_1}{a} \cdot n$.

Proof: Let x be the number of nodes falling in one rectangle with size $c_1 \times a$ and A_n be the event that there is at least one rectangle with more than nodes, by Lemma 2, we get $\Pr(A_n) \leq \frac{\sqrt{n}}{c_1} \times \Pr(x > c_1\sqrt{n}) \leq \frac{\sqrt{n}}{c_1} e^{-c_1\sqrt{n}} (\frac{ec_1\sqrt{n}}{c_1\sqrt{n}})^{2c_1\sqrt{n}} = \frac{\sqrt{n}}{c_1} e^{-c_1\sqrt{n}} (\frac{e}{2})^{2c_1\sqrt{n}}$. It goes to 0 when $n \to 0$. ■

C. Highway System and Related

Most of our routing strategies are built upon the highway system developed in [4]. Here we briefly review its construction and some key properties. To construct a highway system, we partition the square B_a into $m = \frac{a}{\sqrt{2}c}$ cells with a side length c. By appropriately choosing c, we can arrange that the probability that a square contains at least a Poisson point is as high as we want. Here when $a = O(\sqrt{n})$, choosing c as some constant is enough, while when $a = \Omega(\sqrt{n})$, we choose $c = \theta_1 \cdot \frac{a}{\sqrt{n}}$ for some constant θ_1.

Then based on percolation theorem, we can choose c large enough such that with high probability (w.h.p.) there are paths crossing B_a from left to right. These paths can be grouped into disjoint sets of paths: each group has $\lceil \delta \log m \rceil$ paths, crossing a rectangle of width m and height $(\kappa \log m + \epsilon_m)$ cells, for all $k > 0$, δ small enough, and a vanishingly small ϵ_m so that the side length of each rectangle is an integer. Same results still hold when looking for paths crossing B_a from bottom to top. Then by union bound, we claim that there exist both horizontal and vertical disjoint paths *w.h.p.*. These paths are called the *highway system*. From now on, we simply call a node *highway node* if the node is on one of horizontal or vertical (or both) paths, otherwise, it will be called *non-highway node*.

Then we slice the network area into horizontal strips of constant width c_0 such that there are at least as many paths as slices inside each rectangle of size $m \times \kappa \log m + \epsilon_m$ by choosing c_0 appropriately. Then we impose that nodes from the ith slice communicate directly with the ith horizontal path. And it is also proved in [4] that *w.h.p.*, there are at most $\Theta(\sqrt{n})$ nodes contained in each stripe. Finally, we can get the following important lemma.

Lemma 6: [4] The nodes along the highways can achieve *w.h.p.*, a per-flow rate of $\Omega(\frac{1}{\sqrt{n}})$.

III. UNICAST CAPACITY FOR ARBITRARY NETWORKS

Here we study unicast capacity for an arbitrary wireless network. Assume n wireless nodes $\{v_1, v_2, \cdots, v_n\}$ are arbitrarily distributed inside a square B_a with side length a, each node will communicate with constant transmission power P. Here the locations of nodes can be optimally chosen to maximize throughput.

A. When $a = O(1)$

Lemma 7: For an arbitrarily network in B_a with $a = O(1)$, the total unicast capacity for n_s transceiver pairs is $\Theta(1)$.

Proof: The lower bound is clearly $\Omega(1)$ since, in any time slot we pick only one transceiver pair (u and v) to communicate, all other transmitters are silent. In this case, the rate $R(u, v) = \log(1 + \frac{P \cdot \ell(u,v)}{N_0 + I(u,v)}) = \Omega(1)$ because the Euclidean distance $\ell(u, v)$ is at most $\sqrt{2}a$ which is a constant, $I(u, v)$ is zero in this case and N_0 is a constant. Thus, the lower bound of unicast capacity for n unicast sessions is $\Omega(1)$. Clearly, when we pick two nodes within distance $\Theta(a)$, the transport capacity can achieve $\Omega(1 \times a)$.

We then show that the capacity is $O(1)$ by the following observations. Assume for any time slot t, there are $m \geq 2$ simultaneously active links in the network. Then for any transceiver pair u and v, the rate $R(u, v) = \log(1 + \frac{P \cdot \ell(d(u,v))}{N_0 + I(u,v)}) \leq \log(1 + \frac{P \cdot 1}{N_0 + \sum_{m-1} P \cdot 1}) \leq \frac{1}{m-1}$. Since there are m simultaneously active links for time slot t, thus, the total capacity of all network is $O(1)$. Clearly, the upper bound on transport capacity is $\frac{m}{m-1} \cdot a = O(a)$ bits-meters/sec. ■

B. When $a = \Omega(1)$, and $a = O(\sqrt{n})$

We first present the following lemma (see [5] for its proof).

Lemma 8: For any pair of source/destination nodes (u, v), the transport capacity of a direct link $e = (u, v)$ is $\log(1 + \frac{P \cdot \ell(d)}{N_0 + I(u,v)}) \cdot d$ where d is the Euclidean distance between node u and node v. In addition, the transport capacity between u and v will get its maximum value when $d = 1$.

Lemma 9: For arbitrary network, when side length of B_a is $a = \Omega(1)$ and $a = O(\sqrt{n})$, the total unicast capacity for n transceiver pairs is $\Theta(a^2)$. The transport capacity is $\Theta(a^2)$.

Proof: First, we prove that the capacity is at least $\Omega(a^2)$. We partition the whole square into a^2 cells with side length 1. Next, we assume there is one transceiver pair in each cell. Assume for cell S_i, node u and v are chosen as source and receiver respectively. For a cell S_i, we consider the 5 by 5 grid of cells in which S_i is in the centroid of the grid. All other 24 cells are called the nearest neighbor cells of S_i. Based on a TDMA scheduling scheme, we let the transmitter in S_i be able to transmit only if all transmitters in the S_i and S_i's nearest 24 neighbor cells keep silent. Next we show that when all transceiver pairs in all grey cells exchange data simultaneously, for any pair of transmitter u and receiver v, the data rate between them is $\Omega(1)$ due to Lemma 3.

Next, we give a matching upper bound so that our results are indeed tight. First, we partition the whole square region into $\Theta(a^2)$ cells with side length $\Theta(1)$. For any cell S_i, assume there are j simultaneously transmitters $\{v_{i1}, v_{i2}, \cdots, v_{ij}\}$ inside of S_j. Thus the unicast capacity contributed by cell S_i is $\sum_{k=1}^{j} \lambda_{ik}$. Here, λ_{ik} is the feasible transmitting rate of the k^{th} transmitters inside of S_i. Clearly, adding one or more transmitters into S_i or replacing current transmitter(s) with others (originally silent nodes in this time slot) will not improve the unicast capacity contributed by S_i due to Lemma 7. Thus, the total unicast capacity is equal to $\sum_{i=1}^{\Theta(a^2)} \sum_{k=1}^{j} \lambda_{ij}$, which is bounded by $O(a^2 \times 1) = O(a^2)$.

Observe that the total capacity $\Omega(a^2)$ clearly is achievable by carefully placing a pair of nodes with distance $\Theta(1)$ in each cell. This construction also gives us a lower bound $\Omega(a^2)$ on the transport capacity. This finishes the proof. ■

C. When $a = \Omega(\sqrt{n})$

Lemma 10: For arbitrarily network in B_a with $a = \Omega(\sqrt{n})$, the total unicast capacity for n transceiver pairs is $\Theta(n)$.

Proof: Clearly, the upper bound here is $O(n)$ because there are at most $\lfloor \frac{n}{2} \rfloor$ node can transmit simultaneously, and the rate of each pair is at most a constant. Next we show that by the following construction, the unicast capacity can also archive $\Omega(n)$ when side length $a = \Omega(\sqrt{n})$.

We partition the region into $m = \frac{n}{c^2}$ small rectangles with side length $r = c \times \frac{a}{\sqrt{n}}$. Here, we can round c up to some constant such that $c^2 \geq 2$ and m is an integer. In each small square, we put one source/destination pair within small distance d_1 around the center. First, we show when all transmitters transmit simultaneously, for any transceiver pair (u, v), the data rate $R(u, v) = \Omega(1)$ based on a TDMA schedule. The proof idea is exactly same with the one used in Lemma 9. The total interference is

$$I(u, v) \leq \sum_{i=1}^{\infty} 8iP \cdot ((2i-1)\frac{ca}{\sqrt{n}})^{-\beta}$$

Notice that this sum clearly converges if $\beta > 2$ when $a = \Omega(\sqrt{n})$, so $I(u, v)$ is a constant. Thus, the total rate between u and v is $R(u, v) = \log(1 + \frac{P \cdot \ell(d(u,v))}{N_0 + I(u,v)}) = \Omega(1)$, since $\ell(d(u, v)) = \min\{1, |uv|^{-\beta}\}$ is also a constant. Thus, at any time, based on our TDMA scheduling, there are at least $\lfloor \frac{n/2}{9} \rfloor$ links be active simultaneously, so the lower bound capacity for n transceiver pair is $\Omega(n)$. ■

IV. Unicast Capacity for Random Networks

We will study the unicast capacity for random wireless networks based on three scenarios $a = O(1)$, $a = \Omega(1)$ and $a = O(\sqrt{n})$, or $a = \Omega(\sqrt{n})$.

A. When $a = O(1)$

When the side length $a = O(1)$ and n goes to ∞, the unicast case for random wireless networks is similar with the one for arbitrary wireless networks. The following theorem directly follows Lemma 7.

Theorem 11: For a wireless network with randomly placed wireless nodes in a square B_a, the total unicast capacity is $\Theta(1)$ when $a = O(1)$.

Similarly it is not difficult to derive the following theorem.

Theorem 12: For a random wireless network with n randomly placed wireless nodes in a square B_a, the per-flow unicast capacity is $\Theta(\frac{1}{n})$ when $a = O(1)$ and $n_s = n$.

B. When $\Theta(1) \leq a \leq \Theta(\sqrt{n})$

Next we show that when the side length a satisfies $1 \leq a \leq \sqrt{n}$, the capacity for unicast is $\Omega(a)$ by constructing the following routing and link scheduling scheme. By the percolation theory and the results in [4], when we partition the whole square into small cells with side length c, we can select one node from each cell and construct $\Omega(m)$ horizontal and $\Omega(m)$ vertical "highways" (or say disjoint paths) from left to right and from top to bottom respectively as the backbone of the whole wireless network with probability $1 - e^{-nc^2/a^2}$. Here, $m = \frac{a}{c}$, where c is rounded up such that m is an integer. In addition, we can choose c large enough such that $\Omega(m)$ paths can be partitioned into a number of disjoint groups each with $\lceil \delta \log m \rceil$ disjoint paths, and each group are contained in a stripe with width m cells and height $(\kappa \log m - \epsilon_m)$ cells, for all $\kappa > 0$, δ small enough, and a non-zero small ϵ_m such that the side length of each stripe is integer. The same is true when we partition the square into vertical stripes with side length $m \times (\kappa \log m - \epsilon_m)$.

Routing Strategy: Our routing strategy is same as [4]. We briefly review it here. For each pair of source/destination nodes u and v, assume u is in the i^{th} stripe. If u is not on the highway, we will find a highway node u_{en} in the same stripe to be the entrance node of u, *i.e.*, u_{en} will be the first highway node which will relay packets of u. To find this entrance node, we draw a vertical line from u, and the closest highway node (from this line) which is in the same stripe will be chosen as u_{en}. For destination node v, if v is not a highway node, we use the same method to draw a vertical line from v, and find the closest highway node as the exit node u_{ex}.

There are three phases for any pair of source/destination nodes (u, v) to communicate.

1) If u is not the highways nodes, u will find some entrance node u_{en} and send data to u_{en} by one hop.
2) u_{en} will relay the data of u to exit node v_{ex} of node v through highway (involving both vertical and horizontal highways).
3) v_{ex} will transmit the data to v directly at last.

Lemma 13: For any wireless node u, u can achieve a rate of $\Omega(\frac{a}{n})$ to some node v_{ex} on the highway system based on a TDMA schedule when the side length a satisfies $\Theta(1) \leq a \leq \Theta(\sqrt{n})$. Here $\log a > 1$.

Proof: We know that after we partition the whole square into horizontal (or vertical) stripes with size $m \times (\kappa \log m - \epsilon_m)$, node v can find an entrance node u_{en} on one of $\lceil \delta \log m \rceil$ disjoint paths within distance $\kappa \log m + \sqrt{2}c$ by the triangle inequality. By Lemma 3, we can get the data rate between u and u_{en} is $\Omega((\log m)^{-\beta-2})$. In addition, we know there are at most $\log m \times \frac{n}{m^2}$ nodes that will share the bandwidth together due to Lemma 4. Therefore the lower bound of the per-flow capacity is $\Omega(\frac{(\log m)^{-\beta-2}}{\log m \times \frac{n}{m^2}}) = \Omega(\frac{a}{n})$. ∎

The data rate achievable between destination v and the node v_{ex} is $\Omega(\frac{a}{n})$ as well by applying Lemma 13 reversely.

Lemma 14: The nodes on the highways can achieve per-flow capacity rate of $\Omega(\frac{a}{n})$ with high probability based on a TDMA schedule when $\Theta(1) \leq a \leq \Theta(\sqrt{n})$.

Proof: By Lemma 3, because any two adjacent nodes on the highways are at most one cell away and the side-length c is a constant, any two adjacent nodes on highway can communicate with each other with constant rate based on a TDMA schedule.

In addition, if we partition the square B_a into $\frac{a}{c_1}$ stripes with size $c_1 \times a$, each stripe will contain at most $2\frac{c_1 n}{a}$ nodes *w.h.p.*, by Lemma 5. Here, c_1 can be rounded up such that $\frac{a}{c_1}$ is integer. Thus, for each node on the highway, it will relay traffic for at most $2\frac{c_1 n}{a}$ nodes *w.h.p.*. So, the per-flow capacity for each highway node is $\Omega(\frac{a}{n})$. ∎

Based on Lemma 13 and Lemma 14, we have

Theorem 15: The per-flow unicast capacity in random wireless networks in B_a is $\Omega(\frac{a}{n})$ when $\Theta(1) \leq a \leq \Theta(\sqrt{n})$.

Next, by calculating a matching upper bound of per-flow unicast capacity, we can show that our results are indeed tight.

Lemma 16: Given a source/destination pair randomly placed in a square of side length a, the expected Euclidian distance between them is $c_2 a$ for some constant c_2.

Theorem 17: There is a constant c_3 such that, with probability at least $1 - 2e^{-n_s c_3^2/32}$, the data rate that can be supported, for any routing strategy, is at most

$$\frac{c_3 a}{c n_s} = O(\frac{a}{n_s}) \tag{5}$$

Proof: First, we partition B_a into cells with side length c, here c is some constant. Let $C(P_i)$ denote the number of cells a routing path P_i will use, *i.e.*, the number of cells crossed by P_i. Let variable $L = \sum_{i=1}^{n_s} C(P_i)$, denoting the total load of all cells. Here the load of a cell by a routing method is the number of flows visiting the cell for the unicast path constructed. Then $L \geq \sum_{i=1}^{n_s} l_i/(\sqrt{2}\frac{a}{m})$, where l_i denotes the Euclidian distance between the i-th source/destination pair.

We define random variables $X_q = \sum_{j=1}^{q}(l_j - E(l_j))$. Then $E(X_{q+1} \mid X_1, \cdots, X_q) = X_q$. In other words, variables X_i are martingale. In addition, $|X_q - X_{q-1}| = |l_q - E(l_q)| \leq \sqrt{2}a$. From Azuma's Inequality, we have $\mathbf{Pr}(|X_{n_s} - X_0| \geq t) \leq 2\exp(-\frac{t^2}{2\sum_{i=1}^{n_s} 8a^2})$. Let $t = \epsilon \sum_{i=1}^{n_s} E(|l_i|)$. Clearly, $\epsilon n_s c_3 a \leq t \leq \epsilon n_s \sqrt{2}a$ for some constant c_3. Note that $X_0 = 0$. Then, $\mathbf{Pr}(\sum_{i=1}^{n_s} l_i \leq \sum_{i=1}^{n_s} E(l_i) - t) \leq \mathbf{Pr}(|X_{n_s}| \geq t) \leq \exp(-\frac{t^2}{2\sum_{i=1}^{n_s} 8a^2}) \leq \exp(-\frac{(\epsilon n_s c_3 a)^2}{8 n_s a^2}) = \exp(-\frac{n_s \epsilon^2 c_3^2}{8})$ Then, for a constant $\epsilon \in (0,1)$,

$$\mathbf{Pr}\left(\sum_{i=1}^{n_s} l_i \leq (1-\epsilon) n_s \sqrt{2}a\right) \leq 2e^{-\frac{n_s \epsilon^2 c_3^2}{8}}$$

By letting $\epsilon = \frac{1}{2}$, $\mathbf{Pr}\left(\sum_{i=1}^{n_s} l_i \geq n_s \sqrt{2}a/2\right) \geq 1 - 2e^{-n_s c_3^2/32}$. Thus, $\mathbf{Pr}(L \geq n_s m/2) \geq 1 - 2e^{-n_s c_3^2/32}$.

Recall that L denotes the total load of all cells. Then by pigeonhole principle, with probability at least $1 - 2e^{-n_s c_3^2/32}$, there is at least one cell, that will be used by at least $\frac{n_s c_3 m}{m^2}$ flows. According to Lemma 7, we know that the capacity of a cell with constant side length is $O(1)$. Thus, with probability at least $1 - 2e^{-n_s c_3^2/32}$, the data rate that can be supported using any routing strategy, due to the congestion in some cell, is at most $\frac{1}{\frac{n_s m}{c_3 m^2}} = \frac{c_3 m}{n_s} = \frac{c_3 a}{c n_s} = O(\frac{a}{n_s})$ since $m = a/c$ for some constant c. ∎

Obviously, when $a = \sqrt{n}$, our upper bound shows that the results in [4] is indeed tight.

C. When $a = \Omega(\sqrt{n})$

We then address the per-flow unicast capacity when the side length a is of order $\Omega(\sqrt{n})$. Clearly, when we partition square region B_a into cells with side length g, as long as we scale g carefully, the highway system still exists. Notice that here g is not a constant but a function of a and n. Specifically, $g = \theta_3 \frac{a}{\sqrt{n}}$, for some constant θ_3. In this case, we use the same routing strategy as described in Subsection IV-B to give a lower bound of unicast capacity first.

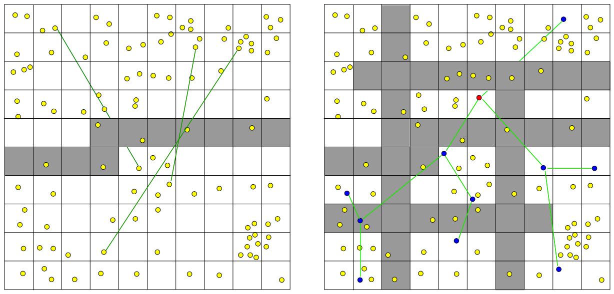
Lemma 18: For a random wireless network with n randomly distributed wireless nodes in a square region with side length a, by our routing strategy, the achievable per-flow unicast rate is $\Omega((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}})$ when $a = \Omega(\sqrt{n})$.

Proof: We partition the whole square into $m^2 = (a/g)^2 = \theta_3^2 n$ cells with side length $g = \theta_3 \frac{a}{\sqrt{n}}$. Here we can choose constant θ_3 carefully such that $\frac{a^2}{g^2}$ is an integer. Then for any cell S_i, the probability that this cell S_i contains at least one node is $1 - e^{-\theta_3^2}$. Again, by appropriately choosing θ_3, we can make the above probability high enough. So the Euclidean distance between any two adjacent nodes (both horizontal and vertical) from the highway system we got by percolation theorem is bounded by $\sqrt{5}g$. Next, we use a TDMA scheduling such that a transmitter inside cell S_i can transmit iff all transmitters inside the closest 24 cells keep silent. Since any two adjacent highway nodes are at most one cell away from each other, then by Lemma 3, the transmission rate between any two adjacent nodes u and v on highway system is at least

$$\Omega(||g||^{-\beta}) = \Omega((\sqrt{5}c_5 \frac{a}{\sqrt{n}})^{-\beta}) = \Omega((\frac{a}{\sqrt{n}})^{-\beta}).$$

In addition, we know that each node on highway relays packets for at most $\frac{2c_1 n}{a} = O(\sqrt{n})$ nodes by Lemma 5. Thus, the per-flow unicast capacity is at least $\Omega((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}})$. ∎

From now on, we will derive matching upper bound on the minimum per-flow unicast capacity. We first give the proof of the existence of a number of cells which contains only a



(a) quasi-closed cut (b) quasi-closed net

Fig. 1. Here a cell is called quasi-closed cell and marked grey if it contains at most c_4 nodes.

constant number of nodes. Then we give an upper bound on minimum data rate by showing the congestion in those cells.

Definition 1: We say a cell is *quasi-closed cell* if it contains at most c_4 nodes, here c_4 is some constant. As illustrated in Figure 1, we call a path of cells *quasi-closed cut* if it contains only quasi-closed cells and crosses from left to right side of B_a. Furthermore, we define the length of a quasi-closed cut as the total number of cells it contains.

See [5] for the proof of the following lemma.

Lemma 19: For any $\frac{5}{6} < p < 1$, there exists a constant c_4 such that the probability that any cell contains no more than c_4 nodes is at least p.

Lemma 20: Some quasi-closed cuts must be crossed by at least $c_5 n_s$ unicast sessions *w.h.p.*., for some constant c_5.

Proof: As shown in [4], for all $\kappa > 0$ and $\frac{5}{6} < p < 1$ satisfying $2 + \kappa \log(6(1-p)) < 0$, there exists a number of disjoint groups containing at least $\lceil \delta \log m \rceil$ disjoint paths in every group, and each group is constraint in a stripe of size $m \times (\kappa \log m - \epsilon_m)$ cells, for δ small enough satisfying

$$\delta \log \frac{p}{1-p} + 1 + \kappa \log(6(1-p)) < 1 \tag{6}$$

and a non-zero small ϵ_m such that the side length of each stripe is integer. Based on Lemma 19, same results can be used to prove the existence of our quasi-closed cuts.

For any constant $\kappa \leq \frac{1}{3} \frac{m}{\log m}$ and $\delta \geq \frac{1}{\log m}$, we pick c_4 carefully based on Lemma 19 to make sure that the preceding inequality (6) is satisfied. Then, *w.h.p.*, there exists at least three disjoint groups containing at least one quasi-closed cut in each group, and every group is bounded by a stripe with width a and height at most $\frac{a}{3}$. Here we only focus on the middle group, for each unicast session, the probability that it must cross the same quasi-close cut in the middle group is no less than $\frac{1}{3}$.

Denote by y the number of unicast sessions which cross the same quasi-closed cut belonging to the middle group. Then $\textbf{Pr}\left(y \leq \frac{n_s}{6}\right) \leq e^{\frac{-2(\frac{n_s}{3} - \frac{n_s}{6})^2}{n_s}}$. Thus, $\textbf{Pr}\left(y > \frac{n_s}{6}\right) > 1 - e^{-\frac{n_s}{18}}$. Here if we set c_5 as $\frac{1}{6}$, the lemma follows when n_s goes to infinity. ■

Lemma 21: With high probability, some of the quasi-closed cells must be crossed by at least $c_6 \frac{n_s}{\sqrt{n}}$ unicast sessions for some constant c_6.

Proof: First, we will prove that, *w.h.p.*. in each group, there exists a quasi-closed cut whose length is no more than $\Theta(\sqrt{n})$. Since there are at least $\lceil \delta \log \frac{a}{g} \rceil$ disjoint paths in each group, and the size of one group is $\frac{a}{g} \times (\kappa \log \frac{a}{g} - \epsilon_{\frac{a}{g}})$, then by pigeonhole principle, there exists at least one quasi-closed cut, say Q, in each group which occupies no more than

$$\frac{\frac{a}{g} \times (\kappa \log \frac{a}{g} - \epsilon_{\frac{a}{g}})}{\lceil \delta \log \frac{a}{g} \rceil} = O(\frac{a}{g}) = O(\sqrt{n})$$

cells, when $g = \Theta(\frac{a}{\sqrt{n}})$. Then together with Lemma 20, there exists at least one cell in Q which is crossed by at least $c_6 \frac{n_s}{\sqrt{n}}$ unicast sessions for some constant c_6. Notice that it equals to $\Theta(\sqrt{n})$ when $n_s = \Theta(n)$. ■

Lemma 22: For a random wireless network with n wireless nodes randomly distributed in a square region with side length a, the minimum per-flow unicast capacity is at most $O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}})$ when $a = \Omega(\sqrt{n})$.

Proof: According to Lemma 21, we know that for any routing strategy, there always exist some cells which contain only constant number of nodes while helping at least $c_6\sqrt{n}$ unicast sessions to relay (when $n_s = \Theta(n)$). Then the per-flow unicast capacity is at most $O(\frac{(\frac{a}{\sqrt{n}})^{-\beta}}{c_6\sqrt{n}}) = O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}})$. ■

Lemma 18 and Lemma 22 together imply Theorem 23.

Theorem 23: For a random wireless network with n randomly distributed wireless nodes in a square region with side length a, the minimum per flow unicast capacity is $\Theta((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{\sqrt{n}})$.

V. MULTICAST CAPACITY FOR RANDOM NETWORKS

To study the multicast capacity, we first present one technique lemma which will be frequently used throughout this section.

Lemma 24: We partition square region B_a into cells with side length g (chosen based on a). Given multicast session M_i, let T_i be the multicast tree for M_i and $C(T_i)$ denote the number of cells the multicast tree T_i will use, then when $k < \theta_4 \cdot \frac{a^2}{g^2}$, *w.h.p.*,

$$C(T_i) \geq \theta_3 \frac{|EMST_{M_i}|}{g}$$

where $|EMST_{M_i}|$ denotes the length of Euclidean Minimum Spanning Tree spanning M_i, θ_3 and θ_4 are some constants.

Proof: We will prove this lemma using some existing results under protocol model, especially the area argument [14]. For the sake of our proof, assume that every node has an artificial "transmission radius" r such that each node v can only communicate with other nodes within its transmission range (a transmitting disk with its center at v and radius r). In addition, we define the area covered by a tree T as the union of the transmitting disks centered at its' nodes. Then by showing a lower bound on the area of the region covered by any multicast tree T, we can give the desired lower bound on the number of cells it will cross.

Recall Lemma 11 in [13], it is proved that in protocol model, the area of the region $D(T)$, *w.h.p.*, is at least $\theta_0\sqrt{k}ar$ when $k < \theta_1 \cdot \frac{a^2}{r^2}$ for some constants θ_0 and θ_1. Here r denotes the transmission range of each node in protocol model and $D(T)$ denotes the region covered by all transmitting disks of all transmitting nodes (internal nodes of T) in any multicast tree T. Unfortunately, this result can not help us directly, since in our model, each node has no fixed transmission range r. Instead, any pair of nodes can communicate with each other even though the data rate may be very small. Based on the original network under Gaussian channel model, we construct a new network under protocol model as follows.

1) Set each node's transmission range as the side length of each cell g, where $g = c$ when $a = O(\sqrt{n})$ and $g = ca/\sqrt{n}$ for some constant c when $a = \Omega(\sqrt{n})$.
2) Add some artificial "additional relay nodes" V_{ad} such that any pair of nodes will have enough relay nodes along its link to make sure that the minimum number of cells the routing path crosses under protocol model is no more than the number of cells the direct link will cross in Gaussian channel model. Notice that V_{ad} cannot be selected as source or receivers, they can only act as relay nodes.

Let T be any multicast tree in the original network under Gaussian channel model and T_p denote the corresponding multicast tree (spanning the same multicast session) constructed on this new network (with additional nodes V_{ad}) under protocol model. We have two important observations here:

1) Our preceding two modifications will not affect the proof for Lemma 11 in [13]. In other words, the lower bound on $|D(T_p)|$ still holds,
2) Furthermore, any link in Gaussian channel model can be simulated by using these artificial "additional relay nodes" in the protocol model such that the number of cells it will cross will not increase. So the lower bound of $C(T)$ is no smaller than the lower bound of $C(T_p)$.

From Lemma 11 in [13], we have, *w.h.p.*, $D(T_p) \geq \theta_0\sqrt{k}ar = \theta_0\sqrt{k}ag$. Since one transmitting disk can cover no more than 4 cells. We have, *w.h.p.*, $C(T_p) \geq \theta_0\sqrt{k}ag/4 \times g^2 = \frac{\theta_0}{4} \cdot \frac{\sqrt{k}a}{g}$. It follows that when $k < \theta_4 \cdot \frac{a^2}{g^2}$, *w.h.p.*, $C(T) \geq \frac{\theta_0}{4} \cdot \frac{\sqrt{k}a}{g}$. Since $|EMST| \leq 2\sqrt{2}\sqrt{k}a$, if we set θ_3 as $\frac{\theta_0}{4}/2\sqrt{2}$, the lemma follows. ■

A. Upper Bound When $a = O(\sqrt{n})$

Here we provide an upper bound of multicast capacity when $a = O(\sqrt{n})$. Similar as previous approach, we partition the square region B_a into cells with side length c where c is some constant, then the total number of cells is $m^2 = \frac{a^2}{c^2} = \Theta(a^2)$.

Lemma 25: [14] Given one multicast session M_i with one source and $k-1$ receivers randomly selected and all receivers are placed in a square B_a, the Euclidean minimum spanning tree $\mathrm{EMST}(M_i)$ has an expected total edge length $c_1\sqrt{k}a$ for a constant $c_1 \in (0, 2\sqrt{2}]$

Theorem 26: When $a = O(\sqrt{n})$, with probability at least $1 - 2e^{-n_s c_8^2/32}$, the per-flow multicast data rate that can be supported using any routing strategy, is at most

$$O(\frac{a}{n_s\sqrt{k}}) \tag{7}$$

Proof: Let variable $L = \sum_{i=1}^{n_s} C(T_i)$, denoting the total load of all cells. Here the load of a cell by a routing method is the number of flows "crossing" the cell for the multicast tree constructed. Then based on Lemma 24, we know that $L \geq \sum_{i=1}^{n_s} \theta_3|\,\mathrm{EMST}(M_i)|/(\frac{a}{m})$ *w.h.p.*. Notice that $E(\sum_{i=1}^{n_s}|\,\mathrm{EMST}(M_i)|) = n_s c_7 a\sqrt{k}$.

Define random variables $X_q = \sum_{j=1}^{q}(|\,\mathrm{EMST}(M_j)| - E(|\,\mathrm{EMST}(M_j)|))$. Then $E(X_{q+1} \mid X_1, \cdots, X_q) = X_q$, so variables X_i are martingale. In addition, $|X_q - X_{q-1}| = ||\,\mathrm{EMST}(M_q)| - E(|\,\mathrm{EMST}(M_q)|)| \leq E(|\,\mathrm{EMST}(M_q)|) \leq 2\sqrt{2}\sqrt{k}a$. Again, from Azuma's Inequality, we have $\mathbf{Pr}(|X_{n_s} - X_0| \geq t) \leq 2\exp(-\frac{t^2}{2\sum_{i=1}^{n_s} 8ka^2})$. Let $t = \epsilon\sum_{i=1}^{n_s} E(\mathrm{EMST}(M_i))$. Clearly, we have $\epsilon n_s c_8\sqrt{k}a \leq t \leq 2\sqrt{2}n_s\epsilon\sqrt{k}a$. Note that $X_0 = 0$. Then,

$$\begin{aligned} &\mathbf{Pr}\left(\sum_{i=1}^{n_s}|\,\mathrm{EMST}(M_i)| \leq \sum_{i=1}^{n_s} E(|\,\mathrm{EMST}(M_i)|) - t\right) \\ \leq\ & \mathbf{Pr}(|X_{n_s}| \geq t) \leq \exp(-\frac{t^2}{2\sum_{i=1}^{n_s} 8ka^2}) \\ \leq\ & \exp(-\frac{(\epsilon n_s c_8\sqrt{k}a)^2}{8n_s ka^2}) = \exp(-\frac{n_s\epsilon^2 c_8^2}{8}) \end{aligned}$$

Letting $\epsilon = \frac{1}{2}$, $\mathbf{Pr}\left(\sum_{i=1}^{n_s}|\,\mathrm{EMST}(M_i)| \geq n_s c_9\sqrt{k}a/2\right) \geq 1 - 2e^{-n_s c_8^2/32}$. Based on Lemma 24, we get $\mathbf{Pr}\left(L \geq n_s\theta_3 c_9\sqrt{k}m/2\right) \geq 1 - 2e^{-n_s c_8^2/32}$. It implies that $\mathbf{Pr}\left(L \geq n_s\theta_3 c_9\sqrt{k}m/2\right) \geq 1 - 2e^{-n_s c_8^2/32}$ if $k \leq \theta_1\sqrt{n}$.

Recall that L denotes the total load of all cells. Then by pigeonhole principle, with probability at least $1 - 2e^{-n_s c_8^2/32}$, there is at least one cell, that will be used by at least $\frac{n_s c_{10}\sqrt{k}m}{m^2}$ flows where $c_{10} = \theta_3 c_9$. Again, according to Lemma 7, the capacity of a cell with constant side length is $O(1)$. Thus, when $n_s = \Theta(n)$, with probability at least $1 - 2e^{-n_s c_8^2/32}$, the per-flow data rate that can be supported is at most, for any routing strategy, $\frac{1}{\frac{n_s c_{10}\sqrt{k}m}{m^2}} = \frac{m}{c_{10}n_s\sqrt{k}} = O(\frac{a}{n_s\sqrt{k}})$. ■

B. Upper Bound When $a = \Omega(\sqrt{n})$

We now give an upper bound for multicast capacity when $a = \Omega(\sqrt{n})$. The main idea is to show the existence of *quasi-closed cell net i.e.*, the cell net which is composed by all quasi-closed cells. Furthermore, by proving that *w.h.p.*, any multicast routing tree will cross a sufficient large number of quasi-closed cells, we can show that some cell will be used by many flows *i.e.*, the congestion in some quasi-closed cells. Please see Figure 1 for illustration.

Next, we explain our proof in details: First we partition the square region B_a into $m^2 = c_8^2 n$ cells with side length $c_8\frac{a}{\sqrt{n}}$ for some constant c_8. Then based on the results in [4] and Lemma 19, we can choose c_8 large enough such that $\Omega(m)$ quasi-closed cuts can be partitioned into a number of disjoint groups each with $\lceil\delta\log m\rceil$ disjoint quasi-closed cuts, and each group is constraint in a stripe of size $m \times (\kappa\log m - \epsilon_m)$, for all $\kappa > 0$, δ small enough, and a non-zero small ϵ_m such that the side length of each stripe is integer. The same is true when we partition the square into vertical stripes with side length $m \times (\kappa\log m - \epsilon_m)$. Notice that all of the horizontal and vertical stripes together partition B_a into *super-cells* with side length $(\kappa\log m - \epsilon_m) \times \frac{a}{m} = (\kappa\log m - \epsilon_m) \times c_8 \cdot \frac{a}{\sqrt{n}}$.

Theorem 27: When $k = O(\frac{n}{\log^2 n})$ and $n_s = \Theta(n)$, with probability at least $1 - 2e^{-n_s c_8^2/32}$, the per-flow data rate that can be supported by any routing strategy is

$$O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{n_s} \cdot \frac{\sqrt{n}}{\sqrt{k}}) \tag{8}$$

Proof: Our proof again is to analyze the load of some cells. We use L to denote the total load of all cells. Then we get $L \geq \sum_{i=1}^{n_s} \theta_3 |EMST(M_i)| / \left((\kappa \log m - \epsilon_m) \times \frac{a}{m}\right)$ based on Lemma 24. Since $\mathbf{Pr}\left(\sum_{i=1}^{n_s} |\,\mathrm{EMST}(M_i)| \geq n_s c_9 \sqrt{k} a/2\right) \geq 1 - 2e^{-n_s c_8^2/32}$, from Lemma 24, we get

$$\mathbf{Pr}\left(L \geq n_s c_{10} \sqrt{k} \frac{m}{\log m}\right) \geq 1 - 2e^{-n_s c_8^2/32}.$$

for some constant $c_{10} = c_9 \theta_3$. Here we use $\mathbb{L}$ to denote the total number of flows crossing some super-cell. Notice that here "crossing" means visiting and leaving. We get

$$\mathbf{Pr}\left(\mathbb{L} \geq L - n_s k = n_s c_{10} \sqrt{k} \frac{m}{\log m}/2 - n_s k\right) \geq 1 - 2e^{-n_s c_8^2/32}$$

We can show that any multicast routing tree will cross at least $\lceil \delta \log m \rceil$ quasi-closed cuts if it crosses three super-cells. Denote by $\mathbb{L}'$ the total number of flows crossing some quasi-closed cut. Then $\mathbb{L}' \geq \frac{\mathbb{L}}{3} \times \lceil \delta \log m \rceil$. It follows that, with probability at least $1 - 2e^{-n_s c_8^2/32}$, the total load of all quasi-closed cell is at least $\frac{n_s c_{10} \sqrt{k} \frac{m}{\log m}/2 - n_s k}{3} \times \lceil \delta \log m \rceil$. Then by pigeonhole principle, with probability at least $1 - 2e^{-n_s c_8^2/32}$, there is at least one quasi-closed cell, that will be used by at least $\frac{\frac{n_s c_{10} \sqrt{k} \frac{m}{\log m}/2 - n_s k}{3} \times \lceil \delta \log m \rceil}{m^2}$ flows, which can be rewritten as $\theta_2 \frac{n_s \sqrt{k}}{\sqrt{n}}$ for some constant θ_2 when $k = O((\frac{m}{\log m})^2)$. Then with probability at least $1 - 2e^{-n_s c_8^2/32}$, the minimum data rate that can be supported using any routing strategy, due to the congestion in some quasi-closed cell, is at most

$$O(\frac{(\frac{a}{\sqrt{n}})^{-\beta}\sqrt{n}}{\theta_2 n_s \sqrt{k}}) = O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{n_s} \cdot \frac{\sqrt{n}}{\sqrt{k}}), \tag{9}$$

when $k = O(\frac{n}{\log^2 n})$. ■

Observe that our result matches the upper bound derived in [6] when $a = \sqrt{n}$. We will derive another upper bound on multicast capacity using different approaches. The basic idea is to show that, for a random network topology, a cluster of nodes exists, that is relatively isolated from the rest of the nodes. Consequently, the average rate of the information that can be sent/received by the nodes of the cluster is limited.

Theorem 28: Consider a random wireless networks, where nodes following a Poisson distribution with parameter $\frac{n}{a^2}$ are distributed in B_a with $a = \Omega(\sqrt{n})$. Assume that n_s random multicast flows are generated. Under Gaussian channel model, the per-flow multicast capacity $\varphi_k(n)$ is at most $O((\frac{a}{\sqrt{n}})^{-\beta}(\log n)^{1-\frac{\beta}{2}}/(n_s p_3))$, which is

$$\varphi_k(n) = \begin{cases} O(\frac{n}{n_s k}(\frac{a}{\sqrt{n}})^{-\beta}(\log n)^{-\frac{\beta}{2}}), & \text{if } k \leq \frac{n}{\log n} \\ O(\frac{1}{n_s}(\frac{a}{\sqrt{n}})^{-\beta}(\log n)^{1-\frac{\beta}{2}}), & \text{if } k \geq \frac{n}{\log n} \end{cases} \tag{10}$$

The proof is similar to the Theorem 2 of [6]. See [5] for proof details. The preceding upper bound on multicast is derived by analyzing an isolated cluster of nodes. For a random wireless network (n nodes randomly distributed in a region B_a, or nodes following a Poisson distribution with parameter $\frac{n}{a^2}$), it is proved in [15] that, *w.h.p.*, the nearest neighbor graph has an edge of length $\Theta(a\sqrt{\frac{\log n}{n}})$. By exploring this long edge, we are able to derive better upper bound:

Theorem 29: Under Gaussian channel model, the per-flow multicast capacity $\varphi_k(n)$ of n_s flows in $2D$ random networks in B_a (for $a = \Omega(\sqrt{n})$), when $k = \omega(\sqrt{n})$, is at most

$$\varphi_k(n) = O\left(\frac{1}{n_s}\frac{n}{k}\left(\frac{a}{\sqrt{n}}\right)^{-\beta}(\log n)^{-\frac{\beta}{2}}\right) \tag{11}$$

Proof: Assume that the longest edge in the nearest neighbor graph of the random network is uv. Then for node v, the probability p_3 that it is chosen as a terminal of a given multicast flow is $p_3 = \frac{k}{n}$. It is easy to show that, with high probability (at least $1 - e^{-frack^2 2n}$), the number of multicast flows that will choose the node v as a terminal is at least $n_s p_3/2$ when $k = \omega(\sqrt{n})$. Observe that the total data rate that node v can receive is at most $R(v) = O\left(\left(\frac{a}{\sqrt{n}}\right)^{-\beta}(\log n)^{-\frac{\beta}{2}}\right)$ since the shortest link incident at node v is at least $\Theta(a\sqrt{\frac{\log n}{n}})$. Then we have $\varphi_k(n) \cdot n_s p_3/2 \leq R(v)$. The theorem then directly follows. ■

Combining Theorem 27, Theorem 28, and Theorem 29, we have the following theorem.

Theorem 30: Under Gaussian channel model, the per-flow multicast capacity $\varphi_k(n)$ of n_s flows in $2D$ random networks in B_a (for $a = \Omega(\sqrt{n})$) is at most

$$\varphi_k(n) = \begin{cases} O((\frac{a}{\sqrt{n}})^{-\beta} \cdot \frac{1}{n_s} \cdot \frac{\sqrt{n}}{\sqrt{k}}) & \text{if } k \leq \frac{n}{(\log n)^\beta} \\ O(\frac{n}{n_s k}(\frac{a}{\sqrt{n}})^{-\beta}(\log n)^{-\frac{\beta}{2}}), & \text{if } k \geq \frac{n}{(\log n)^\beta} \end{cases} \tag{12}$$

When $n_s = n$ and $a = \sqrt{n}$, we have the following corollary.

Corollary 31: Under Gaussian channel model, the per-flow multicast capacity $\varphi_k(n)$ of n flows in $2D$ random networks in $B_{\sqrt{n}}$ is at most

$$\varphi_k(n) = \begin{cases} O(\frac{1}{\sqrt{n}\sqrt{k}}) & \text{if } k \leq \frac{n}{(\log n)^\beta} \\ O(\frac{1}{k}(\log n)^{-\frac{\beta}{2}}), & \text{if } k \geq \frac{n}{(\log n)^\beta} \end{cases} \tag{13}$$

Observe that our upper bound on multicast capacity is achievable when $k = n$ [21] and $k = O(\frac{n}{(\log n)^{2\beta+6}})$ [22]. Our bounds also improve the result in [6].

VI. Literature Reviews

Gupta and Kumar [3] studied the asymptotic capacity of a multi-hop wireless networks for two different models. When each wireless node is capable of transmitting at W bits per second using a fixed range, the throughput obtainable by *each* node for a randomly chosen destination is $\Theta(\frac{W}{\sqrt{n \log n}})$ bits per second under a non-interference protocol, where n in number of nodes. If nodes are optimally assigned and transmission range is optimally chosen, the per-flow throughput is only $\Theta(\frac{W}{\sqrt{n}})$. Similar results also hold for physical interference model. Grossglauser and Tse [8] showed that mobility actually can help to improve the capacity if we allow arbitrary large delay. They show that the average long-term per-flow throughput can be kept constant even as node density increases. Li *et al.* [12] found that the traffic pattern determines whether the per node capacity of a wireless network will scale to large networks. In [7] Gastpar and Vetterli demonstrated the power of network coding: under the point-to-point coding assumption considered in [3], the achievable data rate is constant, independent of the number of nodes. Kyasanur and Vaidya [11] studied the capacity region on random multi-hop multi-radio multi-channel wireless networks.

Broadcast capacity of an arbitrary network has been studied in [10], [16]. They essentially show that the broadcast capacity of a given network is $\Theta(W)$ for single source broadcast and the achievable broadcast capacity per node is only $\Theta(W/n)$ if each of the n nodes will serve as source node. Multicast capacity was recently studied in the literature. Jacquet and Rodolakis [19] claimed that the maximum rate at which a node can transmit multicast data is $O(\frac{W}{kn\log n})$. Recently, serval results [13], [14], [18], [20] were proposed for asymptotic multicast capacity for a large-scale random wireless networks. Assume for each node v_i $(1 \leq i \leq n)$, randomly and independently pick $k-1$ points $p_{i,j}$ $(1 \leq j \leq k-1)$ from the square, and then v_i multicast data to the nearest node for each $p_{i,j}$. They showed the total multicast capacity is $\Theta(\sqrt{\frac{n}{\log n}} \cdot \frac{W}{\sqrt{k}})$ when $k = O(\frac{n}{\log n})$ and when $k = \Omega(\frac{n}{\log n})$, the total multicast capacity is equal to the broadcast capacity, *i.e.*, $\Theta(W)$. Mao *et al.* [20] studied the multicast capacity for hybrid networks. They derived several capacity regimes based on the relations of the number k of receivers per multicast session, the total number n of nodes, and the number m of base stations.

Franceschetti et al. [4] addressed the unicast capacity under Gaussian channel. They proposed a routing and scheduling scheme using highway system based on percolation theorem and proved that a rate $\frac{1}{\sqrt{n}}$ is achievable under Gaussian channel. Zheng [17], [21] pointed out that using multihop relay, the rate of broadcasting continuous stream is $\Theta((\log n)^{-\frac{\beta}{2}})$ in *random extended networks*. Most recently, Li *et al.* [22] proved that, when $n_d = O(\frac{n}{(\log n)^{2\beta+6}})$ and $n_s = \Omega(n^{\frac{1}{2}+\theta})$, the achieving per-session multicast throughput, *w.h.p.*, is of order $\Omega(\frac{\sqrt{n}}{n_s\sqrt{n_d}})$ using *percolation model*, where $\theta > 0$ is a constant. Wang *et al.* [23] recently show that the same results still hold even when $n_d = O(\frac{n}{(\log n)^{\beta}})$. All the above results are derived under the *bounded propagation model* ([24]) and for a single network. Recently, Keshavarz-Haddad and Riedi [6] derived some upper bounds on multicast capacity for Gaussian channel model. They also present algorithms for multicast and claimed that the capacity achieved by their method matches the upper bound. Their bounds are not tight, *e.g.*, rate of W is not achievable when $k \geq \frac{n}{\log n}$.

VII. Conclusions

In this paper, we studied the unicast and multicast capacity for wireless networks under Gaussian channel model when nodes are deployed in a square region B_a with side-length a. We derived asymptotic matching upper-bounds and lower-bounds of unicast capacity for arbitrary and random wireless networks in different cases. Our results close the gap for unicast capacity [4] for example, when $a = \sqrt{n}$. We also present new upper-bounds on multicast capacity for random networks where nodes follow poisson distribution. Our upper-bounds improve the previous result and use new analyzing techniques. A number of interesting and challenging questions remain as future work. A main challenging question is to close the gap on multicast capacity by presenting possibly new tight upper-bounds and designing algorithms to achieve the asymptotic multicast capacity.

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